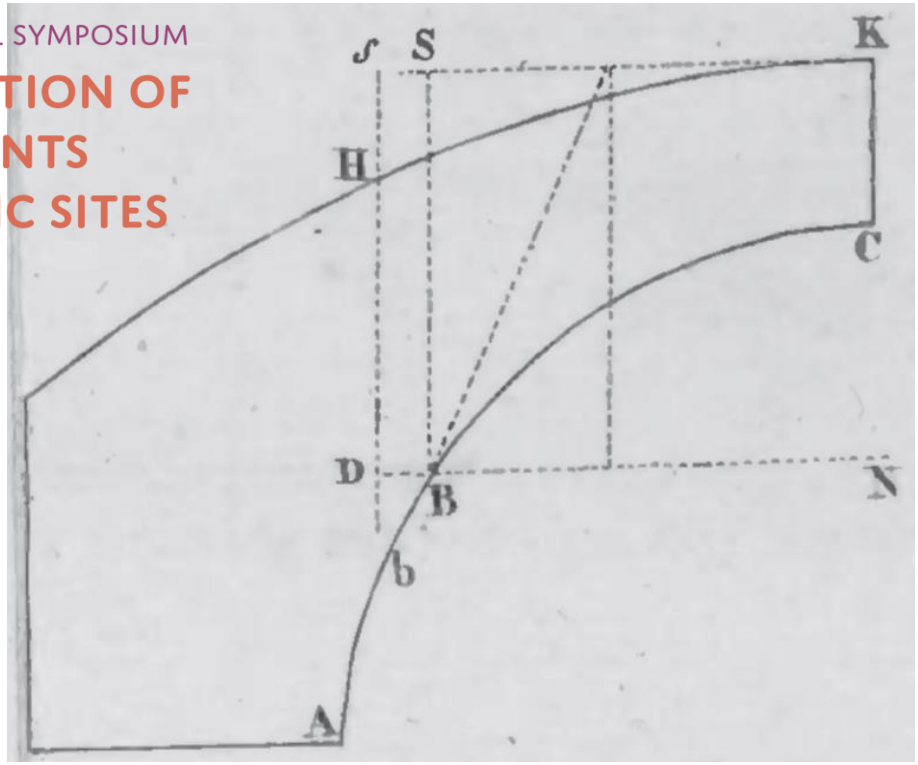
INTERNATIONAL SYMPOSIUM
PRESERVATION OF MONUMENTS & HISTORIC SITES
ATHENS 2026 TC301



From Hooke's "Hanging Chain" and Milankovitch's "Druckkurve" to a Variational Formulation: The Adventure of the Thrust Line of Masonry Arches

Nicos Makris
Professor of Structures and Applied Mechanics
Department of Civil and Environmental Engineering
Southern Methodist University
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The Reconstruction of the North-East Corner of the Temple of Zeus at Nemea: 2003-2009










N
© 2015 Google
Google earth







Semicircular masonry arches are known from the ancient times—before the Roman arch



Olympia Stadium, Greece



Nemea Stadium, Greece



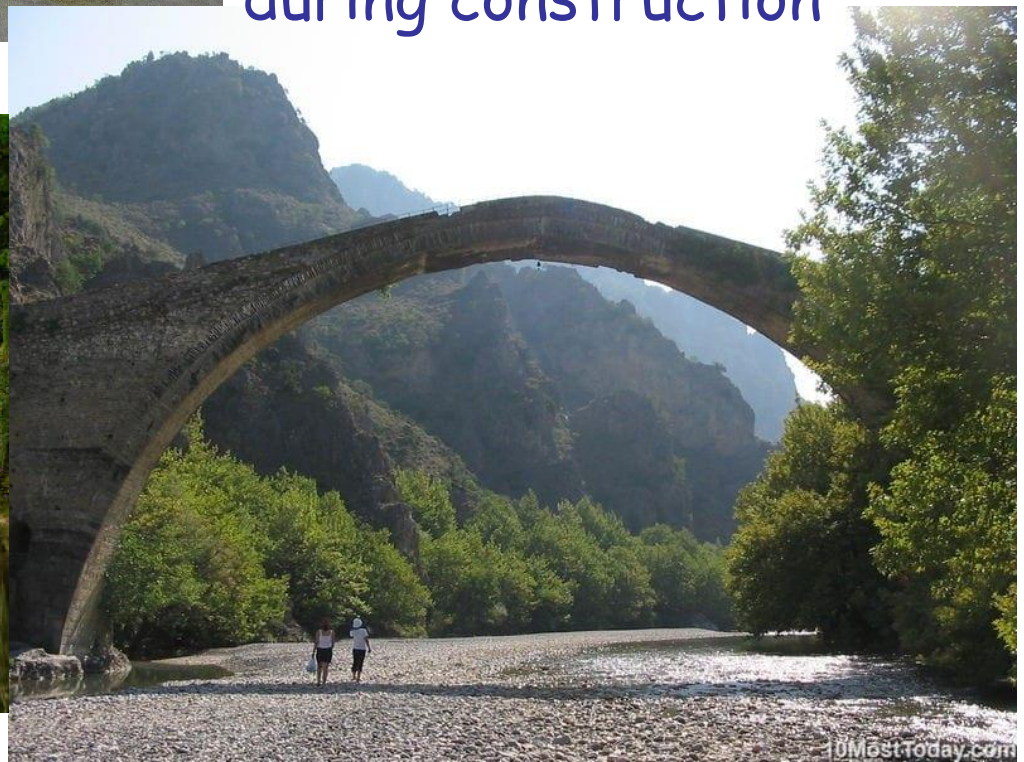
Theatre of Filippi, Kavala, Greece

Given the long history of these structures their exact stress and deformation state is not known—a situation that challenges any kind of stress analysis. On the other hand, a limit-state analysis is independent of the existing stress and deformation state.

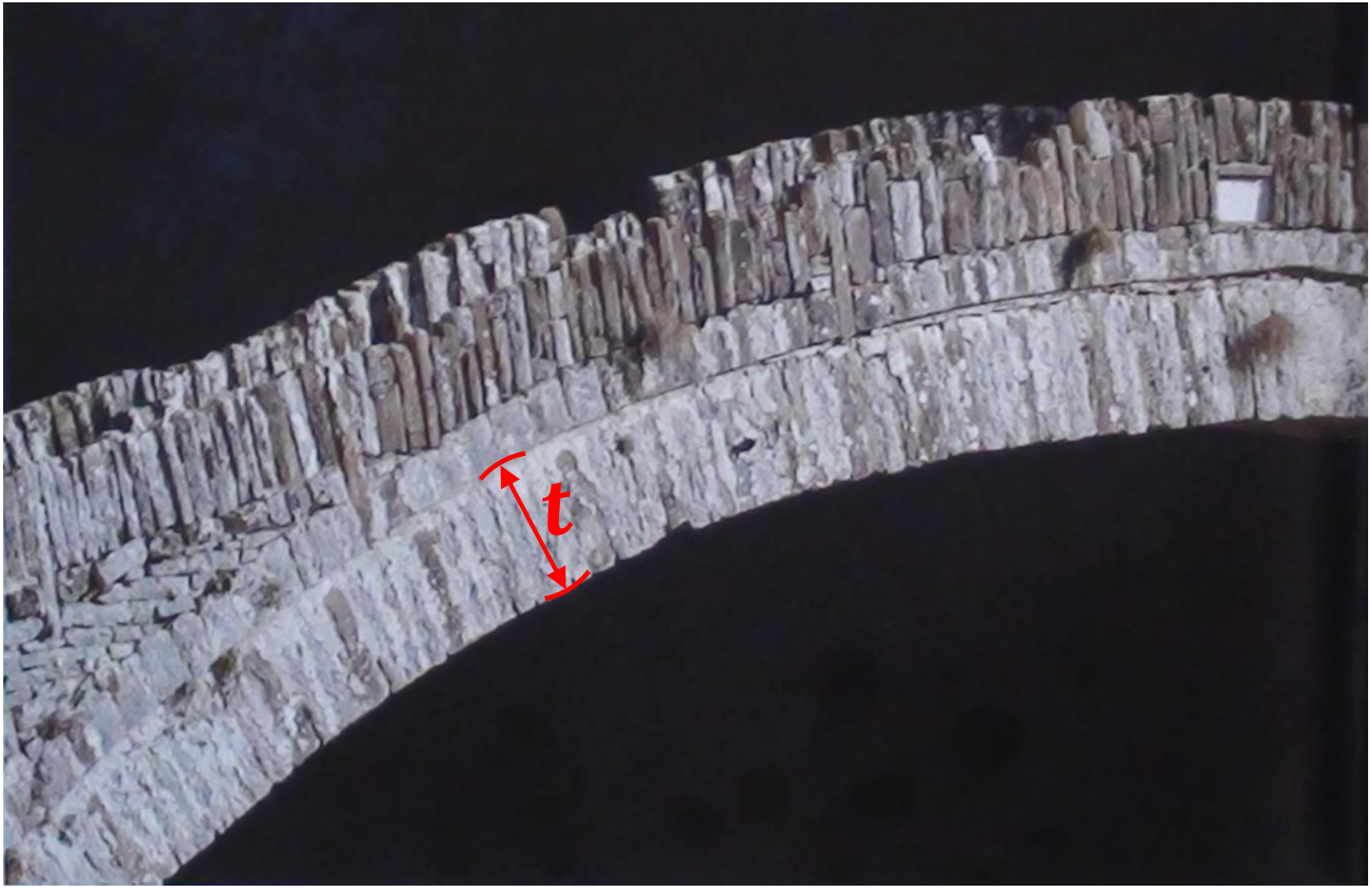


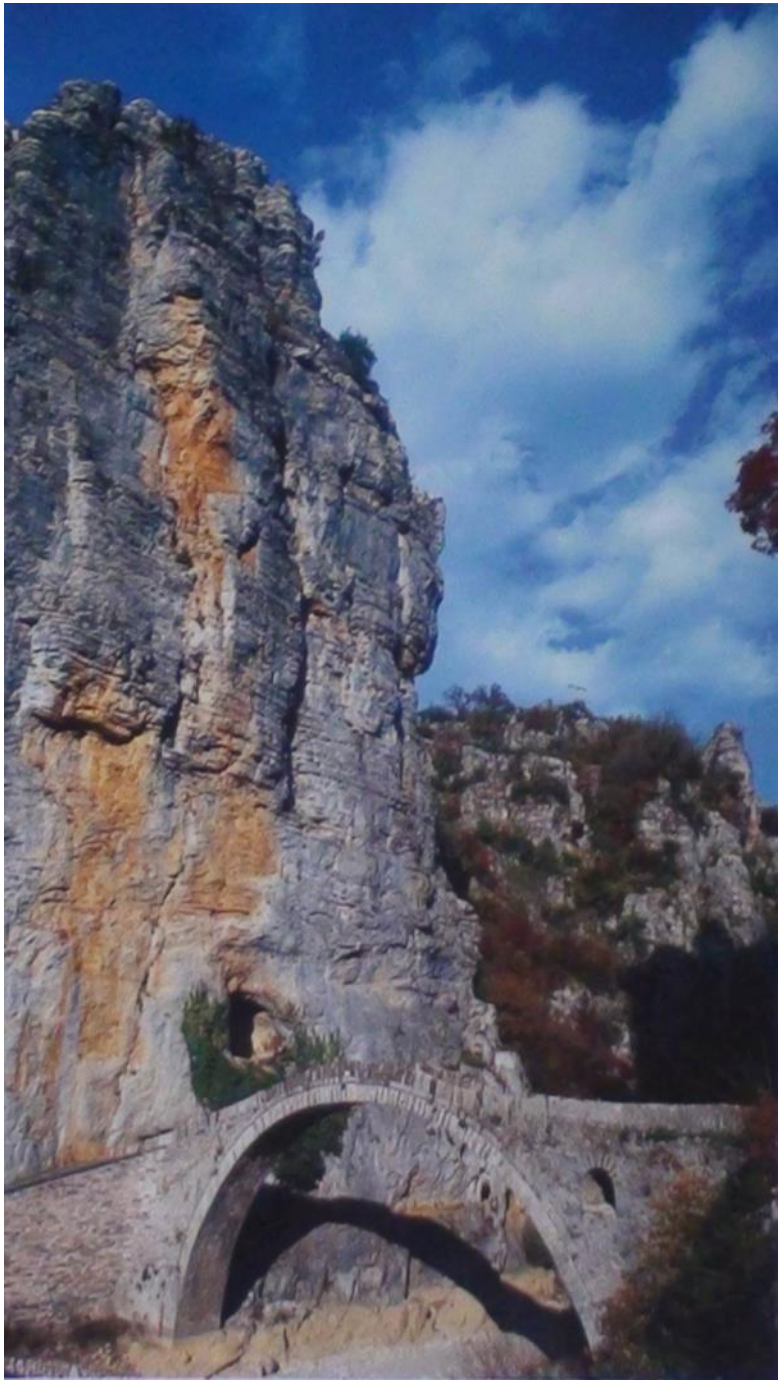
The **challenge of sizing the thickness of stone arches** continues to be in the forefront of structural engineering through the centuries given the **immense human effort** needed during construction





The challenge of sizing the thickness of stone arches emerged as one of the most important optimization problem in the 18th and 19th centuries.





The main question:

What is the minimum value of t/R, for a masonry arch to carry its own weight?

The Noutsos-Kokkoros stone arch bridge at the Bikos canyon (1750-1768)



A

DESCRIPTION

OF

HELIOSCOPES,

And some other

INSTRUMENTS

MADE BY

ROBERT HOOKE,
Fellow of the *Royal Society.*

Hos ego, &c.

Sic vos non vobis——.

Bibliothek des Polytechn.

LONDON,

Printed by *T. R.* for *John e Martyn* Printer to the *Royal Society*, at the *Bell* in St. *Pauls* Church-yard, 1676.

any Character or Language imaginable, as well for Philosophical as for common and constant use. And I have this further to desire of my Reader, who will be at the pains to decipher and understand this description, that he would only make use of it for his own information, and not communicate the explication thereof to any that hath not had the same curiosity with himself.

This I do, not so much to hinder the spreading of this description here delivered, as to revive, and, if possible, bring into use and practice that excellent Design: It being a Character and Language perfectly free from all manner of ambiguity, and yet the most copious, expressive and significative of any thing or Notion imaginable, and, which recommends it most to common use, the most easie to be understood and learnt in the World. *See Table the third.*

To fill the vacancy of the ensuing page, I have here added a *decimate* of the *centesme* of the Inventions I intend to publish, though possibly not in the same order, but as I can get opportunity and leasure; most of which, I hope, will be as useful to Mankind, as they are yet unknown and new.

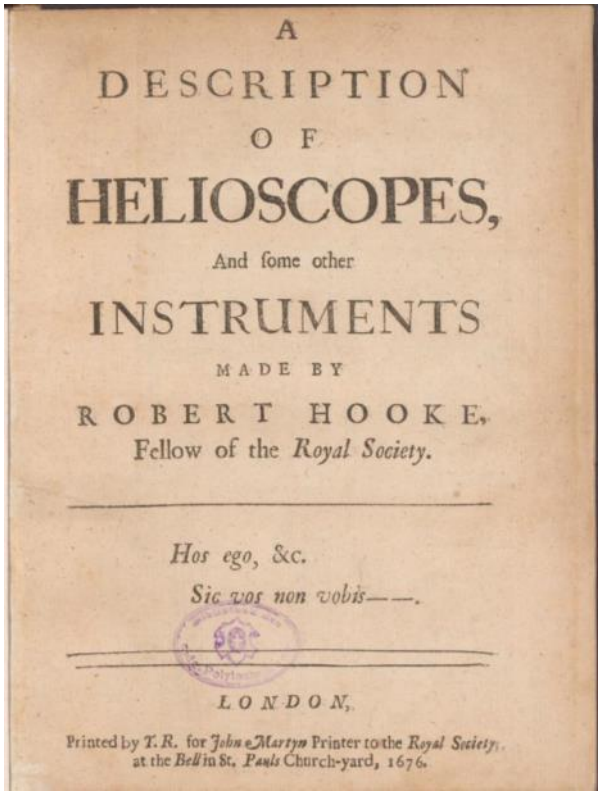
1. *A way of Regulating all sorts of* Watches *or* Time-keepers, *so as to make any way to equalize, if not exceed the* Pendulum-Clocks *now used.*

2. *The true Mathematical and Mechanichal form of all manner of* Arches *for Building, with the true butment necessary to each of them.* A Problem which no *Architectonick* Writer hath ever yet attempted, much less performed. abccc ddeeeee f gg iiiiiii llmmmmnnnnnoopr r ssstttttttuuuuuuuux.

3. *The true Theory of* Elasticity *or* Springiness, *and a particular Explication thereof in several Subjects in which it is to be found: And the way of computing the velocity of Bodies moved by them.* ceiiinosssttuu.

4. *A very plain and practical way of counterpoising Liquors, of great use in* Hydraulicks. Discovered.

5. *A new sort of Object-Glasses for* Telescopes *and* Microscopes, *much outdoing any yet used.* Discovered.

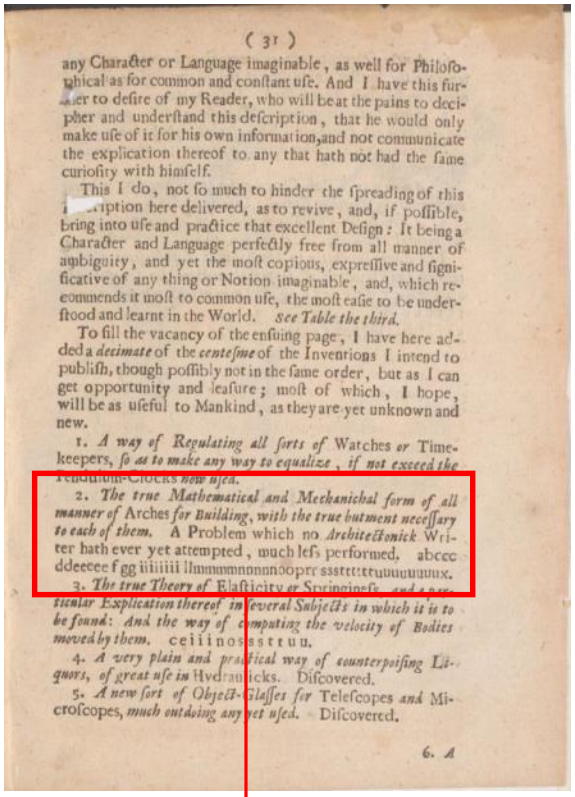


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(31)

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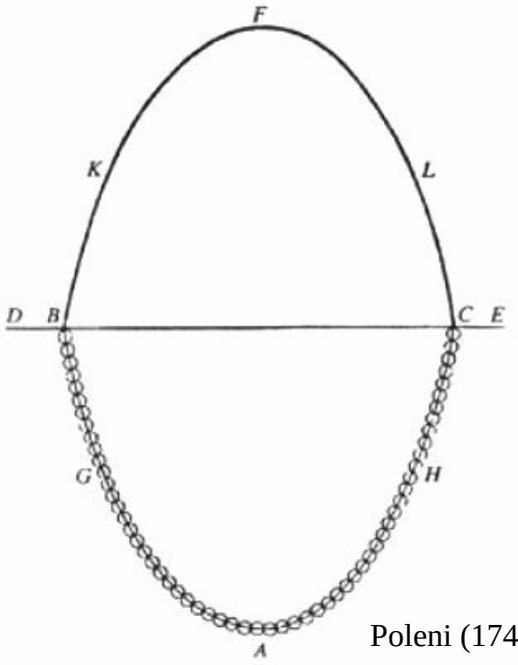
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6. *A*

Robert Hooke (1676)



Poleni (1748)

abcccddeeeeefggiiiiiiiillmmmmnnnnnooprrsssttttttuuuuuuuuvx

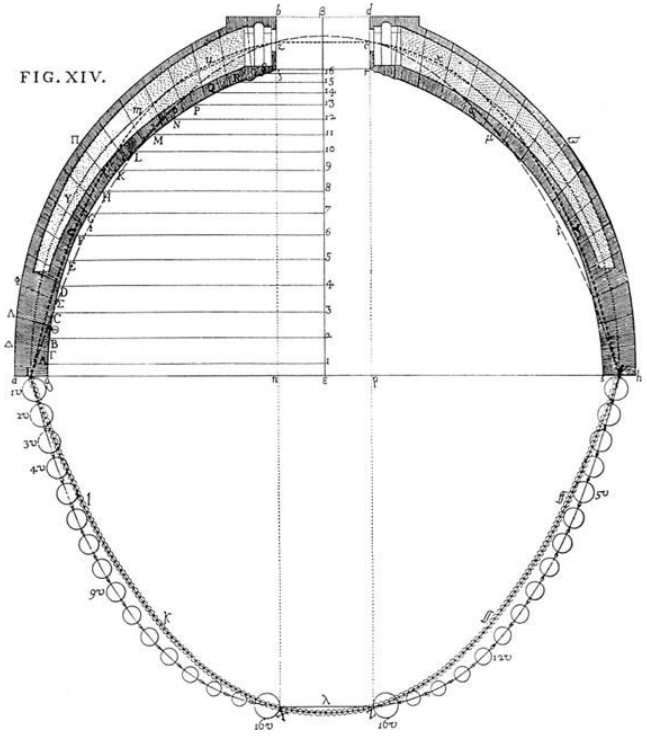
2 years after Hooke's death (1703), the executor of his will uncovered the latin anagram:

Ut pendet continuum flexile, sic stabit contiguum rigidum inversum

As hangs the flexible line, so but inverted will stand the rigid arch

Remarkable application of Hooke's "hanging chain" on the stability analysis of arches and domes

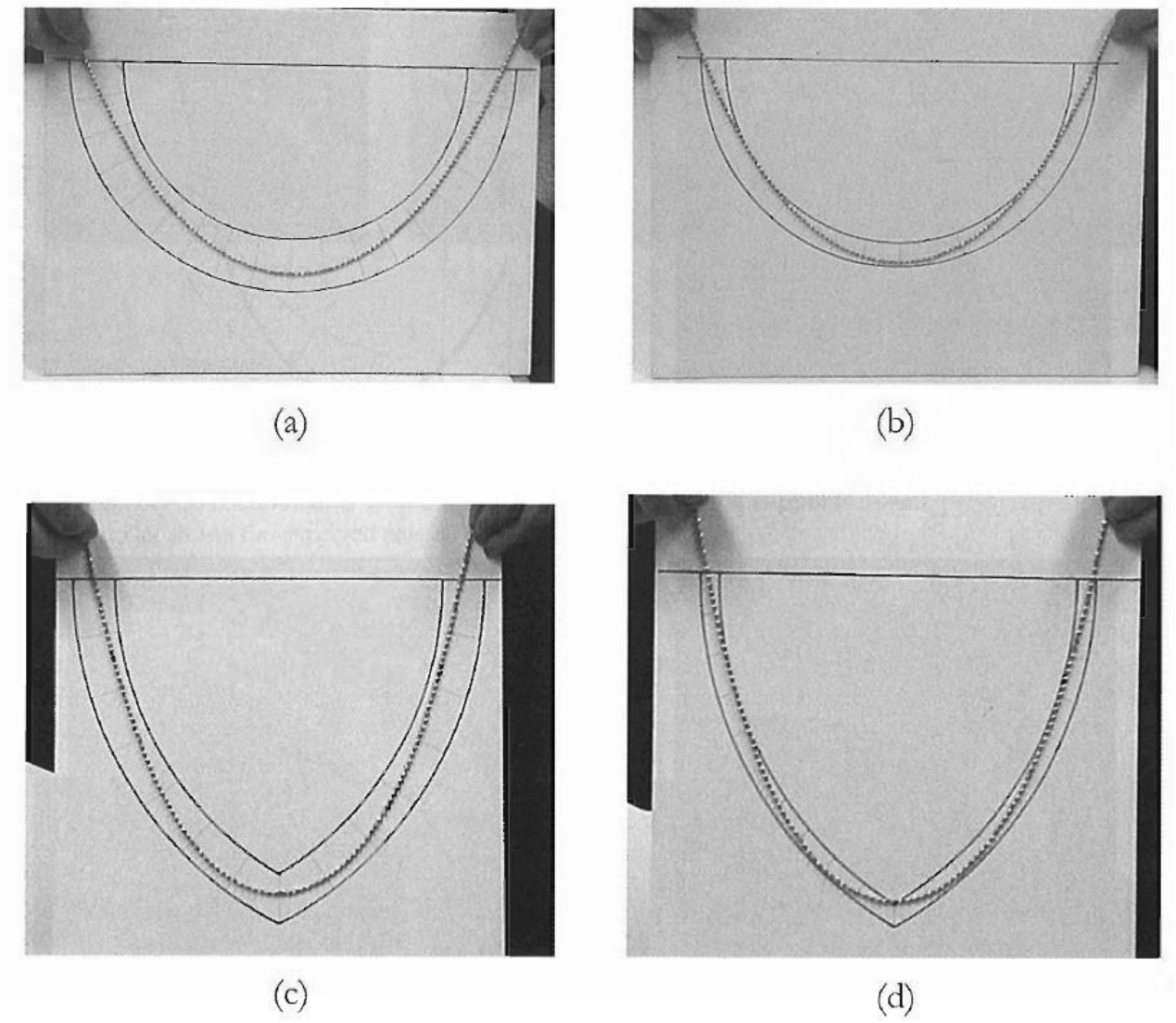
Poleni (1748) assessed the safety of the cracked dome of Saint Peter's Church.



Antonio Gaudi (1852-1926) design the Church of Colonia Güell with the help of a 3D hanging chain model



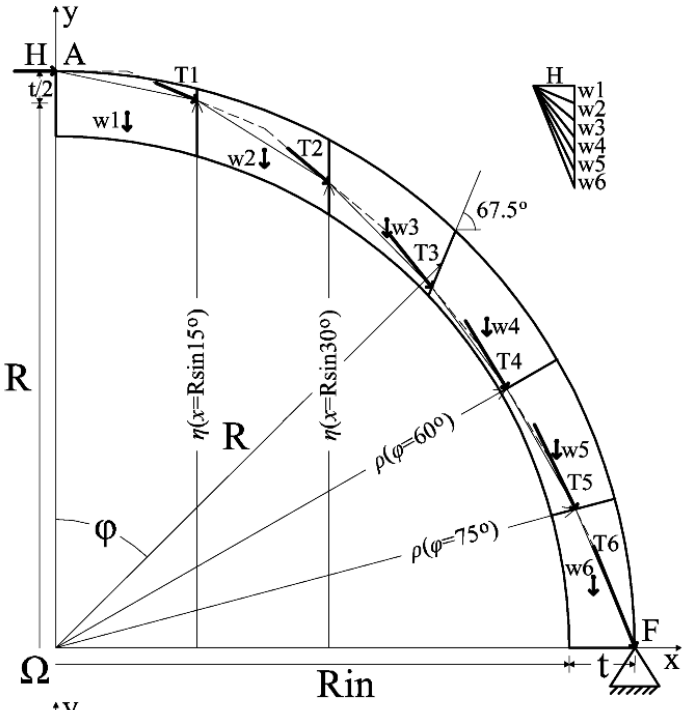
The remarkable directness of Hooke's "hanging chain" is probably the reason that it is widely believed that the catenary (the alysoid) is a physically admissible **thrust-line** of the masonry arch

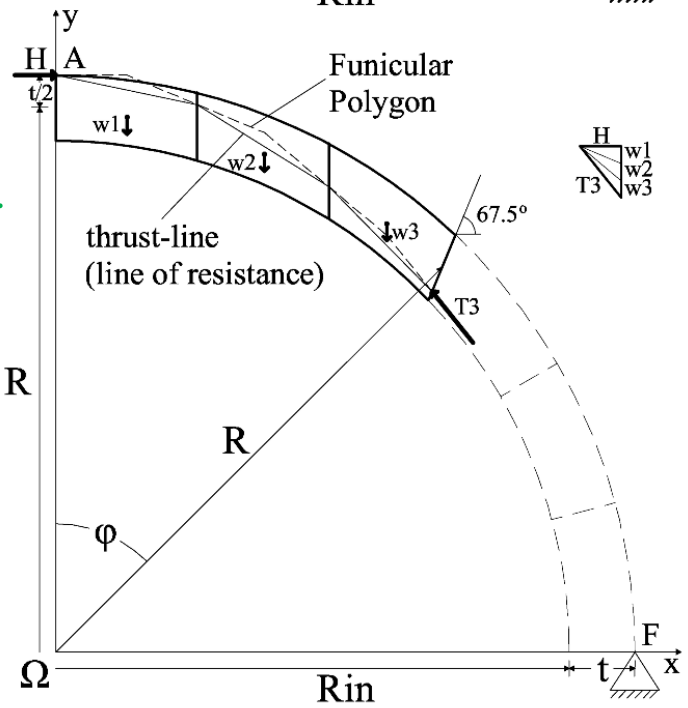


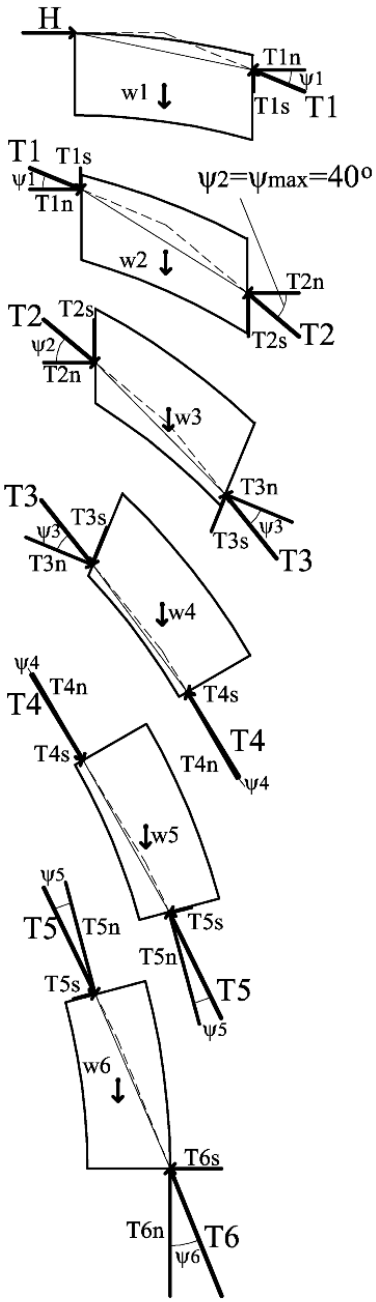
From Huerta (2006)

Thrust-line (line of resistance, Moseley 1843, or druckkurve, Milankovitch, 1904, 1907):

Is defined as the geometrical locus of the application points of the resultant thrust-force that develops at any cross-section of the arch.



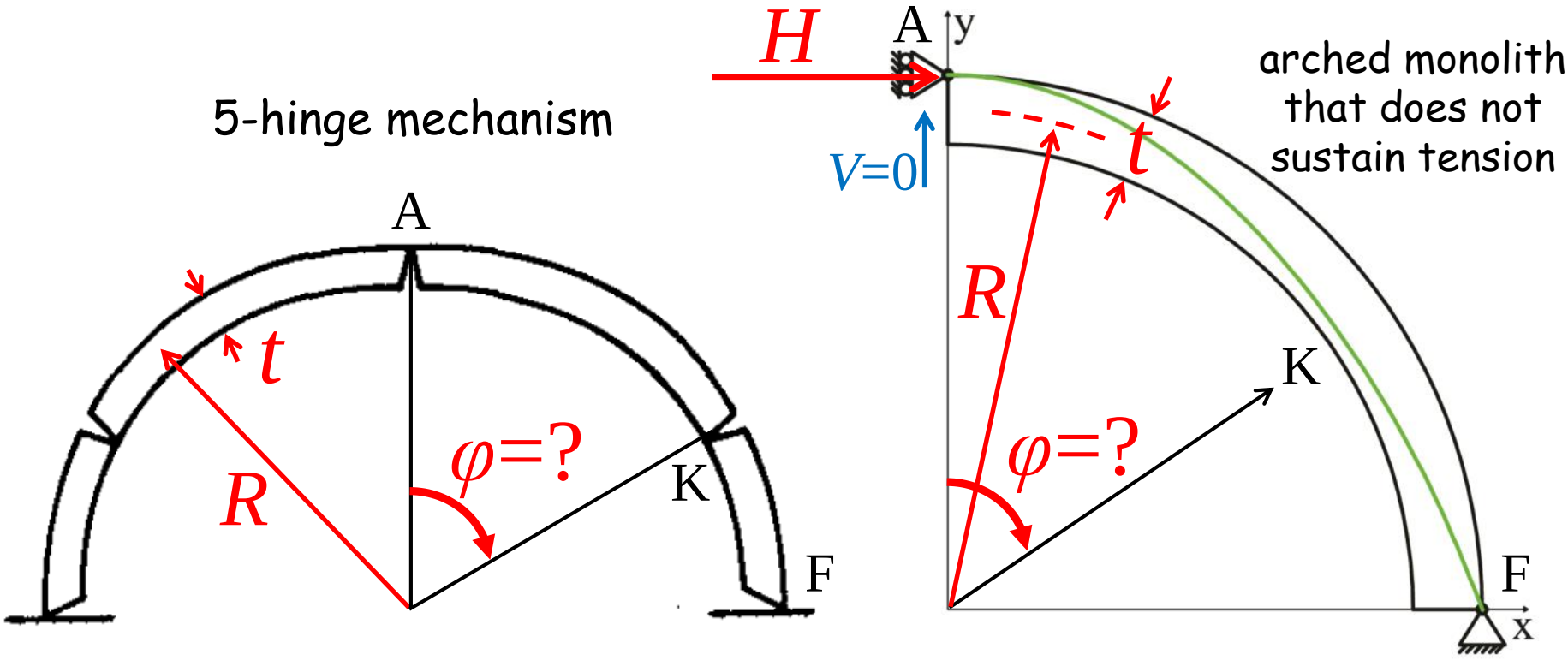




Limit-state of symmetric masonry arches

Statement of the problem: Pierre Couplet (1729, 1730)

Couplet, P. De la poussée des voûtes, *Histoire de l'Académie Royale des Sciences*, Paris (1729, 1730)



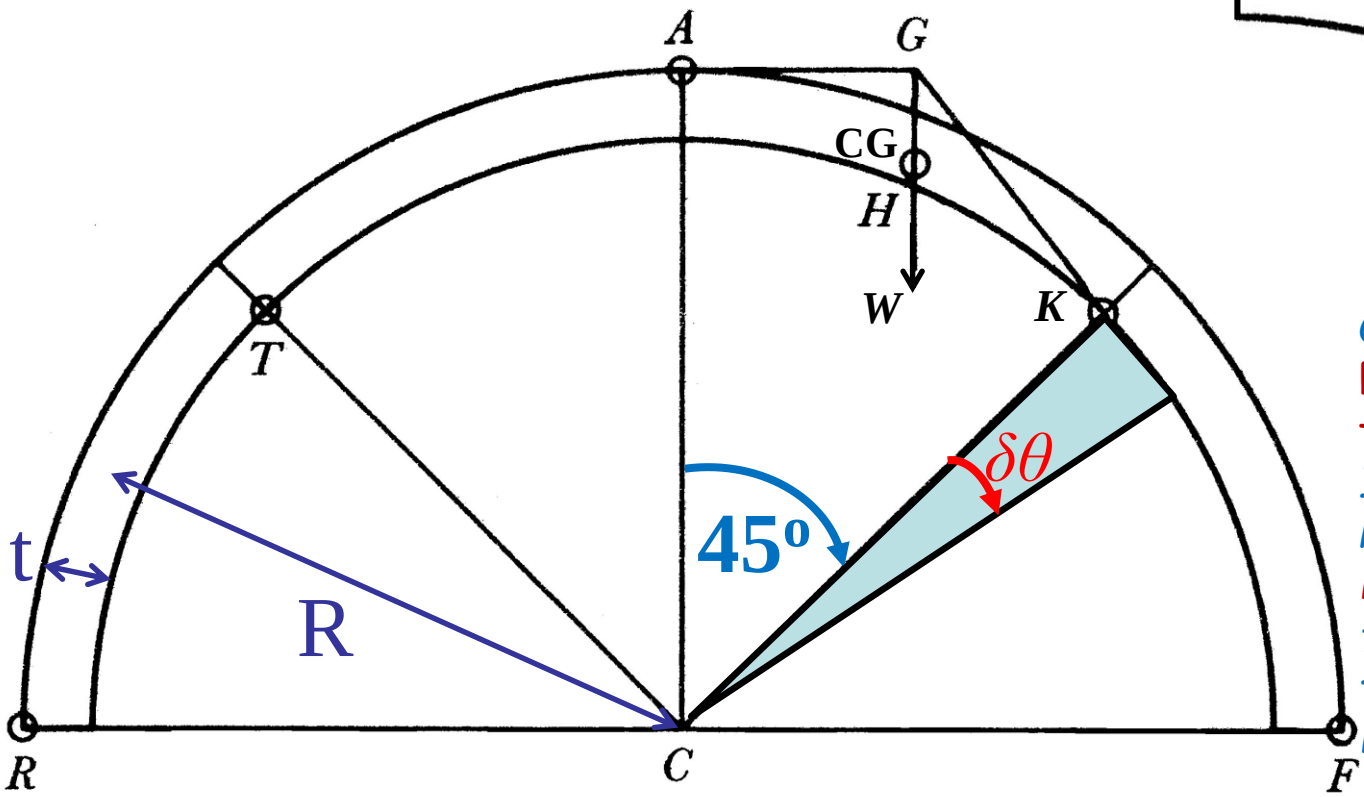
3 unknowns: H, t/R, φ

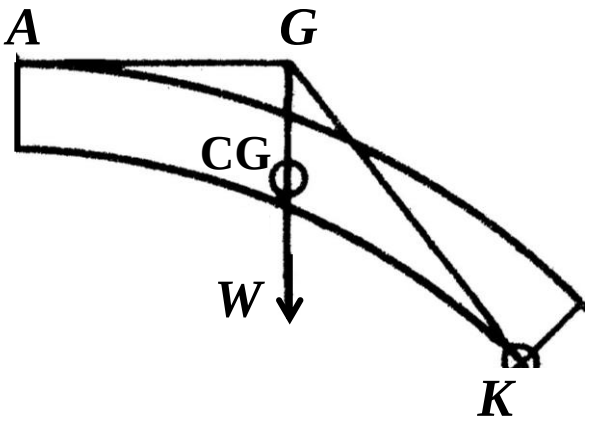
2 equations: $\Sigma M_F=0$ and $\Sigma M_K=0$

Minimum thickness of the semicircular arch

Couplet (1729,1730)

t/R=0.101, (φ=45º)





Clearly GK is shown to be NOT be tangent to the intrados

In several occasions P. Couplet has been incorrectly criticized for NOT taking GK tangent to the intrados

Pierre Couplet, by assuming the rupture angle **φ=45º**, he forced the selection of the mechanism; therefore, his solution is **unconservative**, given that a neighboring mechanism might have developed at a larger thickness

Homage on Pierre Couplet's remarkable contributions

At the dawn of the twentieth century, the Serbian engineer and astronomer **Milutin Milankovitch** showed in a lucid way that when a masonry arch forms **a radial rupture**, **what is tangent at the intrados hinge K is the thrust line—not the thrust force** at the location of the rupture. Unfortunately, Milankovitch's seminal work in 1904-1907 remained nearly unknown until very recently; and we only now recognize that **Pierre Couplet's** intuition for **not taking the resultant thrust force, *GK*, tangent to the intrados was indeed correct**!

So **in the absence of energy methods** (**they were formalized in a useful way a century later by E. Clapeyron**), back in 1730, **Pierre Couplet did as well as one could possibly do** to estimate the minimum thickness of a semicircular masonry arch subjected to its own weight (***t/R* = 0.101**).

State of knowledge in the 18th century

1696	Johann Bernoulli, *Brachistochrone curve*
1729, 1730	Pierre Couplet: *Minimum thickness of arches*
1733	Leonhard Euler, *Elementa calculi variationum*
1762, 1773	Joseph Louis Lagrange (1736-1813), *Euler-Lagrange equation*
1773	Charles-Augustin de Coulomb (1736-1806), *Theory of "minima & maxima"*
1785	*L. Mascheroni, Early statements of the principle of stationary potential energy*: ***Mascheroni, L.: Nuove ricerche sull'equilibrio delle volte. Bergamo (1785)***

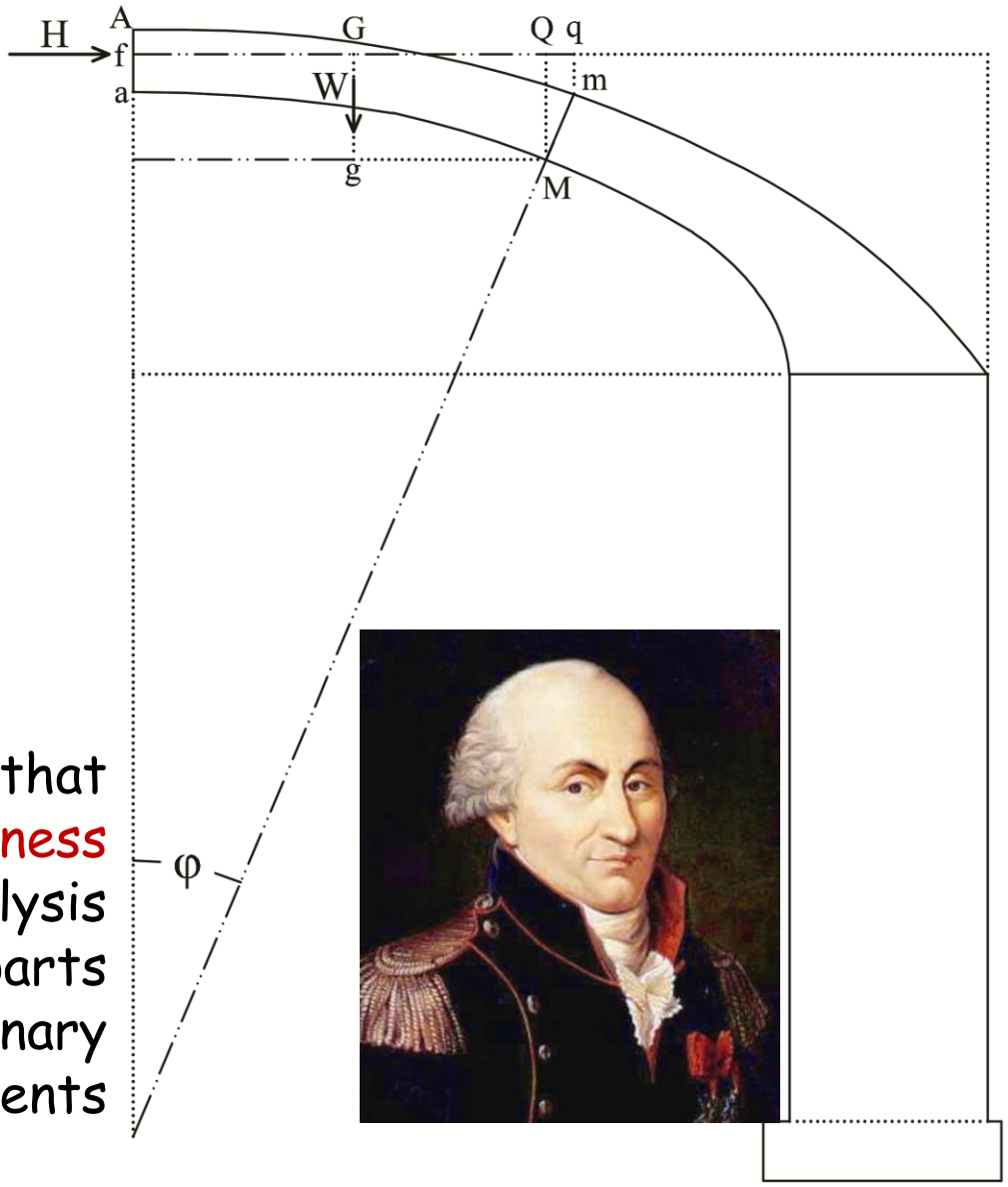
Charles-Augustin de Coulomb
Theory of Maxima and Minima, 1773

$$W\frac{\overline{gM}}{\overline{QM}} < H < W\frac{\overline{Gq}}{\overline{qm}}$$

$$\max H_{\min} \le H \le \min H_{\max}$$

Higher of the lower bound

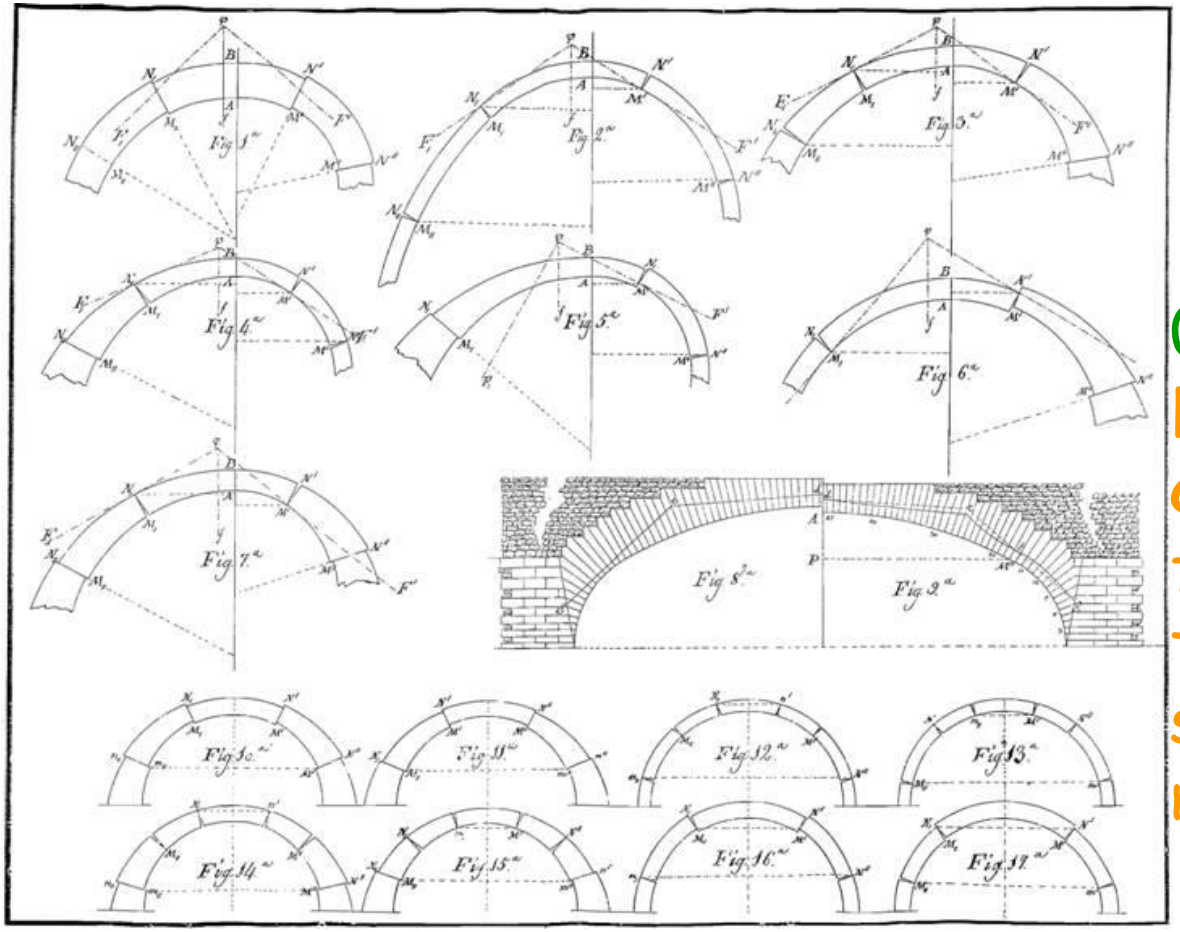
Lower of the upper bound



Augustin Coulomb recognizes that masonry arches have finite thickness and his minima and maxima analysis from moment equilibrium departs from the concept of the catenary (alysoid) along which bending moments are invariable zero!

Joaquín Monasterio, Spanish engineer (1805-1810)

Nueva Teórica sobre el Empuje de Bóvedas



$1/9<t/R_{in}<1/8$

$R_{in}=R-t/2$

$0.1053<t/R<0.1176$

First known **correct result** for the minimum thickness of the semicircular masonry arch

Albuerne & Huerta (2010)

Important Remark:

Monasterio's lower limit t/R=0.1053 is **LARGER** than Pierre Couplet's unconservative solution t/R=0.101

Contribution on the stability (limit state) analysis of masonry arches and bridges

France: late 17th, 18th and early 19th century: The idea of **Collapse Hinge-Mechanism**

P. de LaHire (1640-1718): First application of funicular polygon to the solution of arch problem: 1695: book "Traité de Méchanique"

P. Couplet (1729,1730): First attempt to calculate the minimum thickness of masonry arches

A.F. Frézier quotes tests by **Danizy** (1732) describes clear modes of collapse
1739: book "Traité de la Coupe des Pierres"



C. A. de Coulomb (1736-1806): 1773: theory of "minima & maxima"
Essai sur une application des règles de maximis et minimis a quelques problèmes de statique, relatifs à l'architecture. Mémoires de Mathématique et de Physique présentés à l'Academie Royale des Sciences

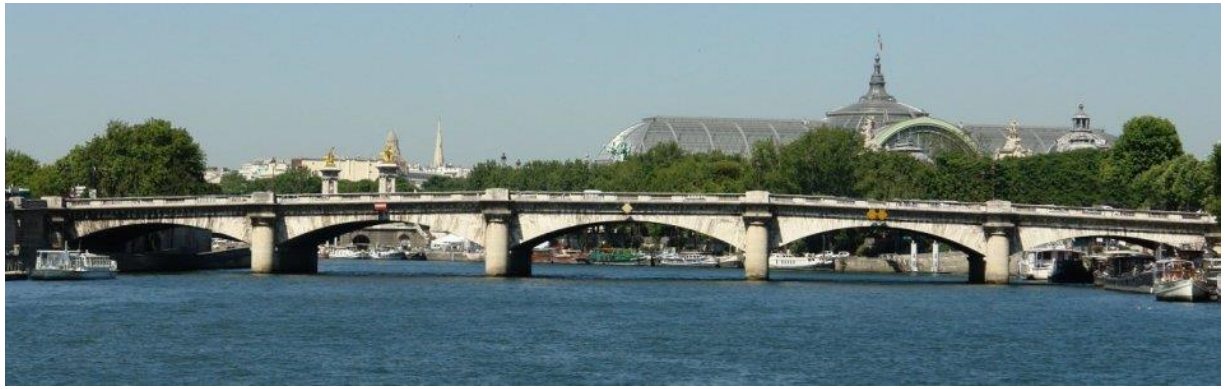
Considerable number of tests at the end of 18th century by
Gauthey, Boistard, Rondelet

Jean-Rodolphe Perronet (1708-1794): Designed and constructed 14 major masonry arch bridges in Paris within a period of 40 years.

G. Lamé and E. Clapeyron (1823). Mémoire sur la stabilité des voûtes, *Annales des Mines*, 8.

- 1750–1760 – Bridge at Orléans
- 1757–1765 – Bridge at Mantes
- 1758–1764 – Bridge at Trilport
- 1765–1786 – Bridge at Château-Thierry
- 1766–1769 – Pont Saint-Edne at Nogent
- 1768–1774 – Bridge at Neuilly-sur-Seine
- 1770–1771 – Pont Les Fontaines
- 1774–1785 – Bridge at Sainte-Maxence sur l'Oise
- 1775 – Bridge at Biais-Bicheret
- 1776–1791 – Bridge at Nemours
- 1776- – Bridge at Mirepoix, Ariege
- 1784–1787 – Bridge at Brunoy
- 1786–1787 – Bridge at Rosoy
- 1786–1791 – Pont Louis XVI, later renamed Pont de la Concorde, Paris





Jean-Rodolphe Perronet (1708-1794)

In order to achieve all this monumental work in **Structural Engineering**, Perronet developed unique skills in **construction management** (on which he wrote extensively) and he is widely recognized as the **Founder of Construction Management**

As the first director, organizer and professor of the École des Ponts et Chaussées, **Jean-Rodolphe Perronet** was a true spiritual father to his students and introduced new teaching and administration methods which seem very contemporary to modern eyes.

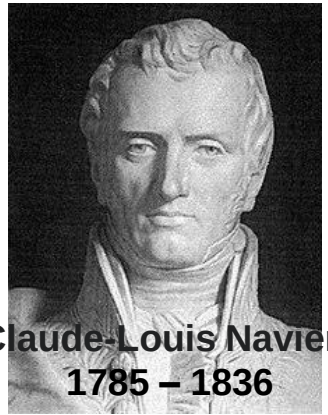
The Beginning of Modern Civil Engineering

1747: Bureau des élèves des ponts et chaussées

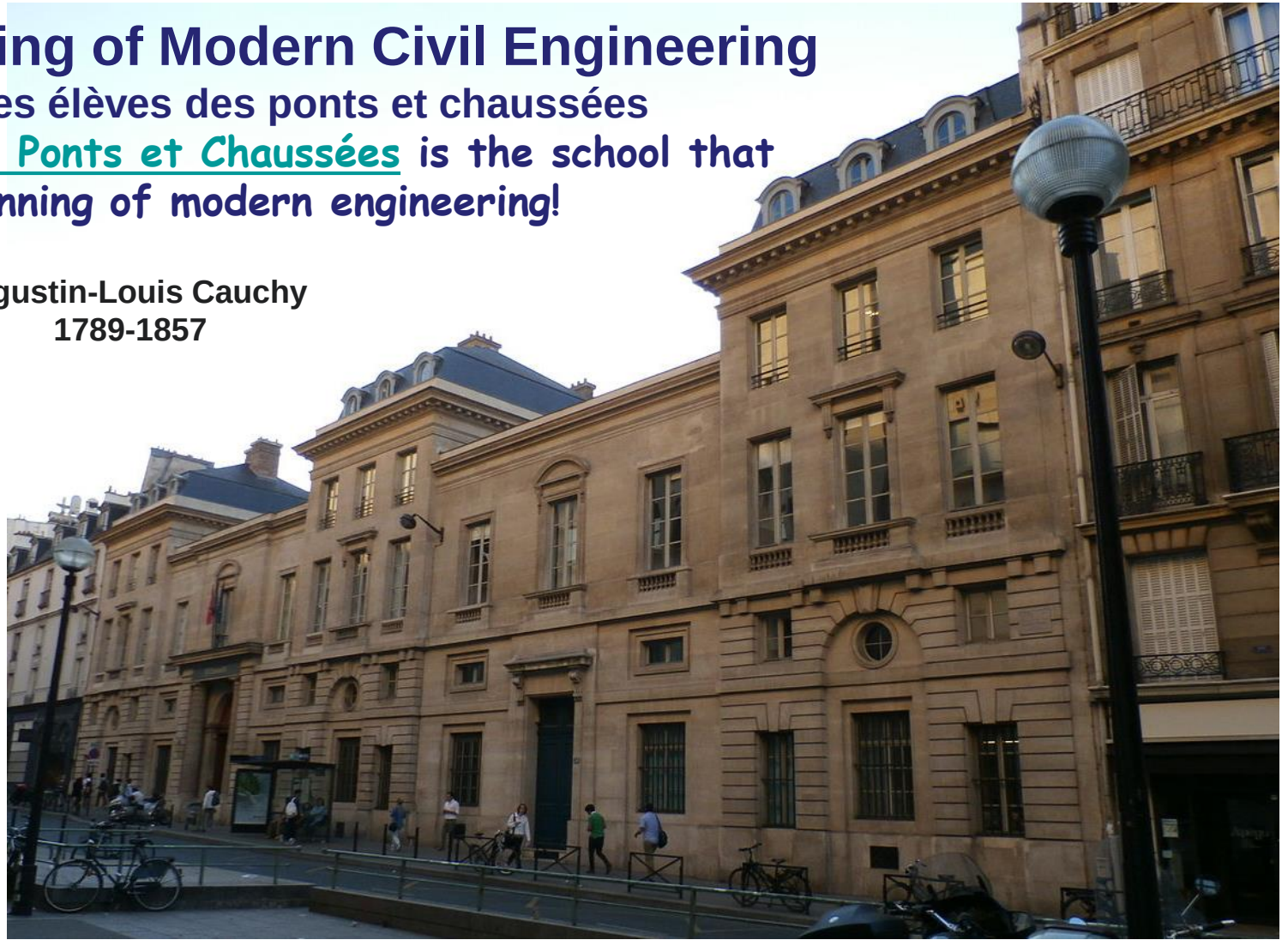
1775: École des Ponts et Chaussées is the school that marks the beginning of modern engineering!



Augustin-Louis Cauchy
1789-1857

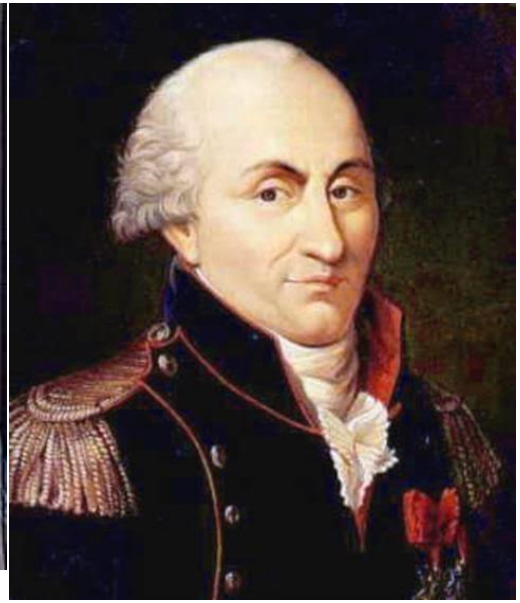


Claude-Louis Navier
1785 – 1836





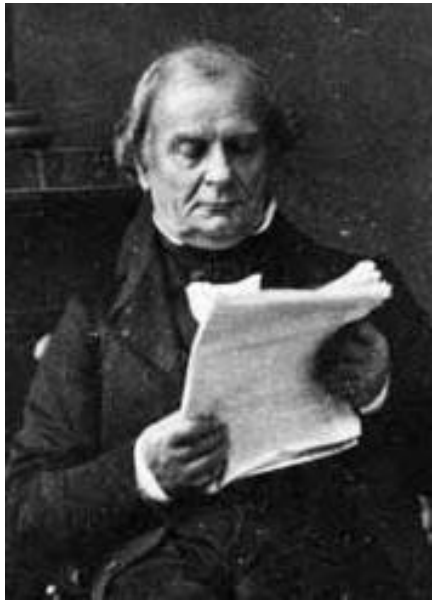
Jean-Rodolphe Perronet (8 October 1708 – 27 February 1794) was an eminent French structural engineer and educator known for designing and planning the construction of many major stone arch bridges in Paris His best-known work is the Pont de la Concorde (1787).



Charles-Augustin de Coulomb: 14 June 1736 – 23 August 1806, was a French officer, engineer, and physicist. He is best known as the discoverer of what is now called Coulomb's law—that is the description of the electrostatic force of attraction and repulsion. He also did important work on friction and failure of soil, creating the basis for the later development of of soil mechanics.

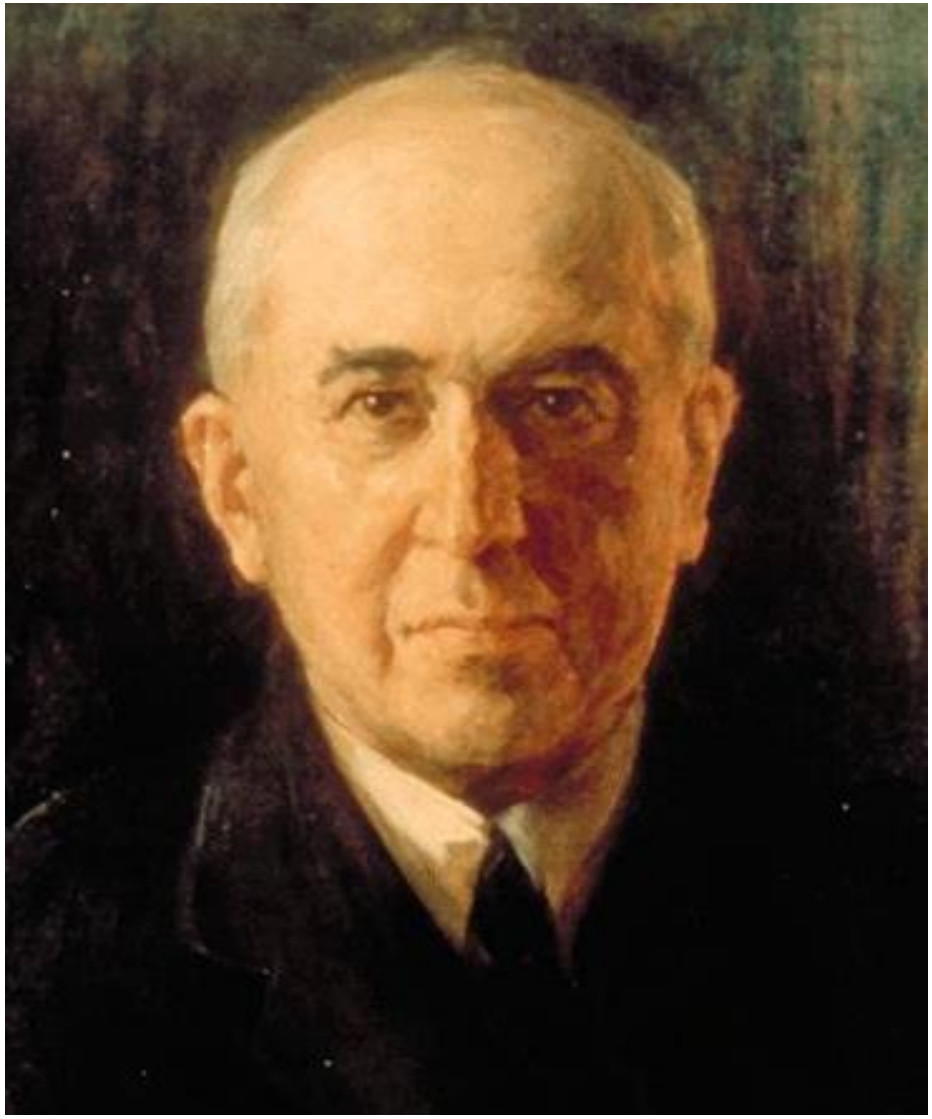


Gabriel Lamé: 22 July 1795 – 1 May 1870, was a French mathematician who contributed to the theory of partial differential equations by the use of curvilinear coordinates, and the mathematical theory of elasticity—including a finite strain theory upon developing elaborate mathematical tools.



Benoît Paul Émile Clapeyron: 26 January 1799 – 28 January 1864) was a French engineer and physicist, one of the founders of thermodynamics. He is known for developing the ideal gas law, while he formalized the concept of internal strain energy and work-energy balance in elastic bodies creating the basis of energy methods in Structural Mechanics

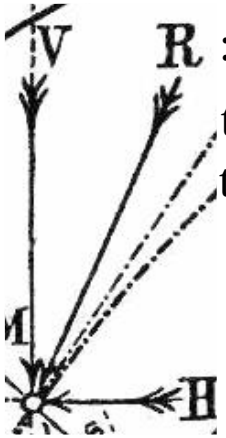
Milutin Milankovitch (1904, 1907)
"Theorie der Druckkurve"



Milutin Milanković (when anglicized Milutin Milankovitch, 28 May 1879 – 12 December 1958) was a Serbian mathematician, astronomer, climatologist, geophysicist, and **civil/structural engineer**

Milanković developed the **"Canon of the Earth's Insolation"**, which characterizes the climates of all the planets of the Solar System and **subsequently explaned Earth's long-term climate changes caused by changes in the position of the Earth in comparison to the Sun, now known as Milankovitch cycles.** This partly explained the ice ages occurring in the geological past of the Earth, as well as the climate changes on the Earth which can be expected in the future.

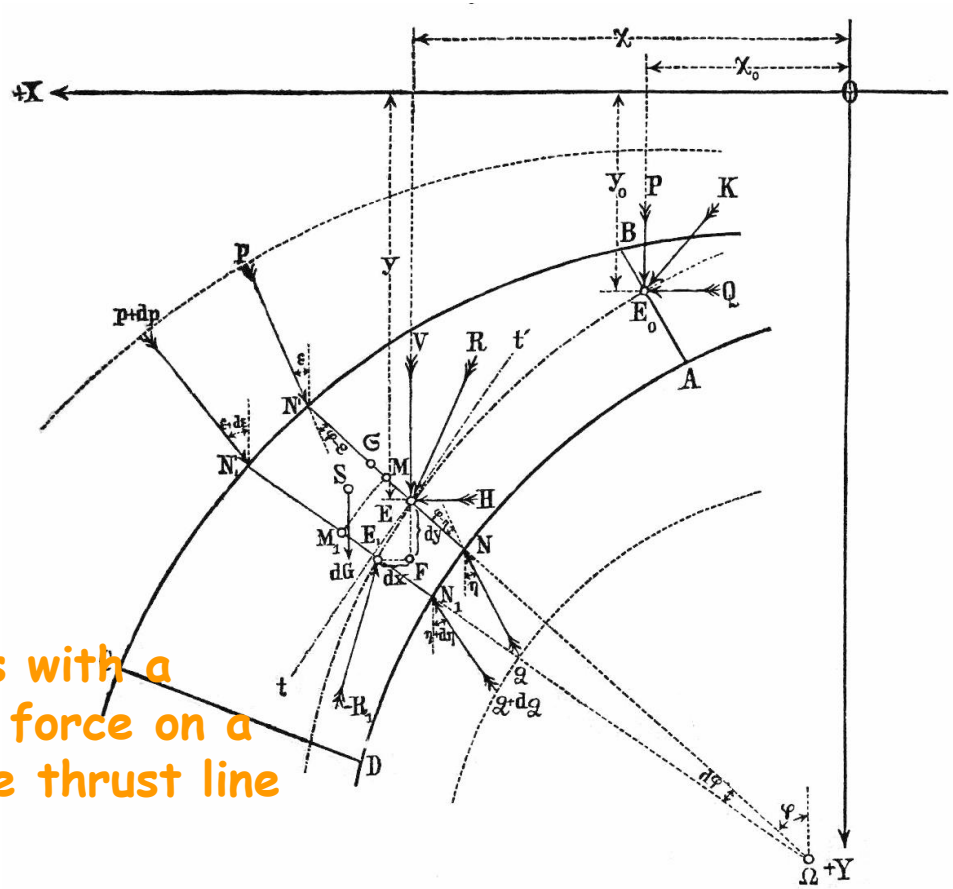
Difference between the **thrust-force on a radial cut** and the **tangent to the thrust-line**



R : thrust-force on a radial cut

tangent to the thrust-line

thrust-line

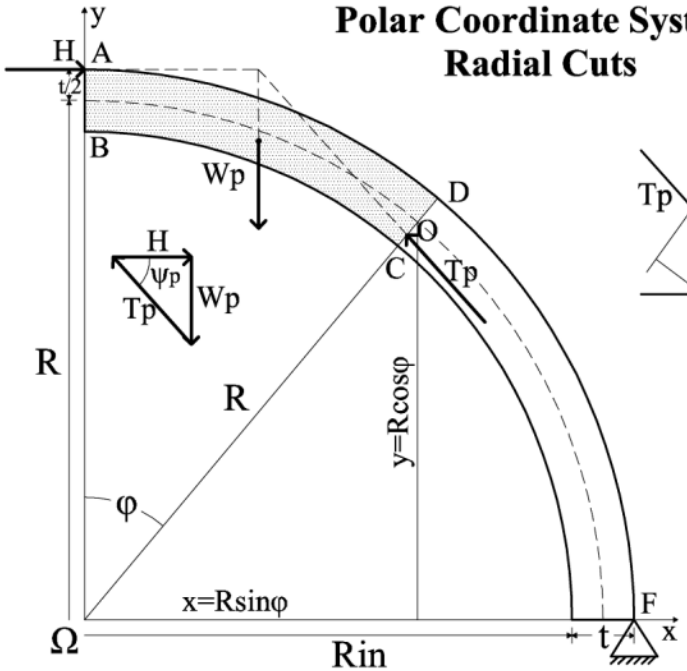


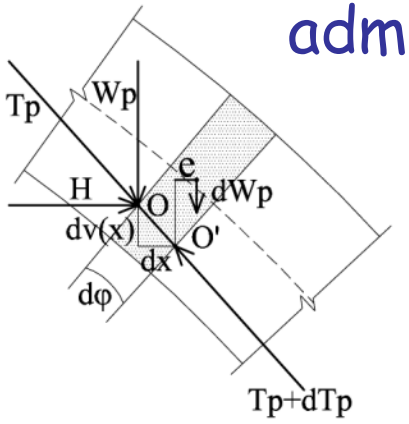
Milutin Milankovitch's analysis starts with a clear understanding that the thrust force on a radial section is NOT tangent to the thrust line

General expression for the difference between the direction of the thrust-force and the tangent to the thrust-line for arches with variable section

$$\frac{V}{H}-\frac{dy}{dx}=\frac{1}{H}\left[g\beta\delta\rho\left(\frac{\delta^2}{12\rho}+\xi\right)\sin\varphi\frac{d\varphi}{dx}+p\left(\frac{\delta}{2}+\xi\right)\sin(\varphi-\varepsilon)\frac{de}{dx}+q\left(\frac{\delta}{2}-\xi\right)\sin(\varphi-\eta)\frac{di}{dx}\right]$$

Construction of a physically admissible thrust-line



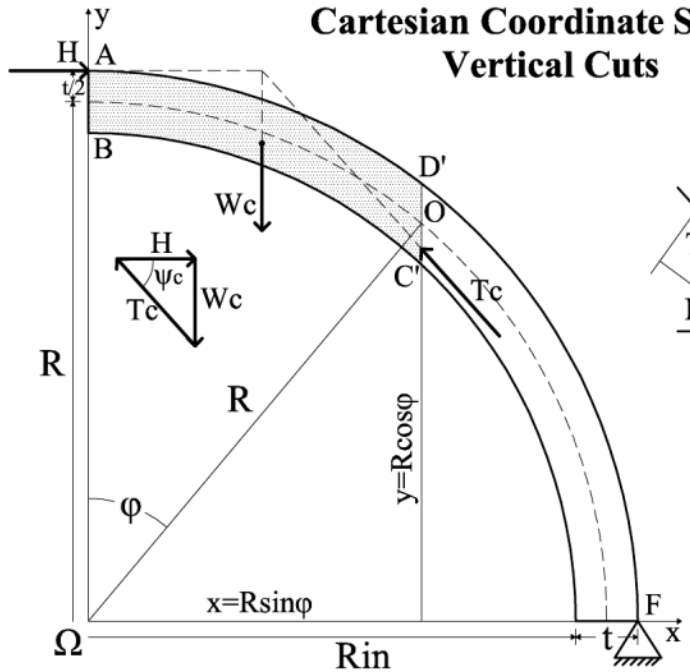


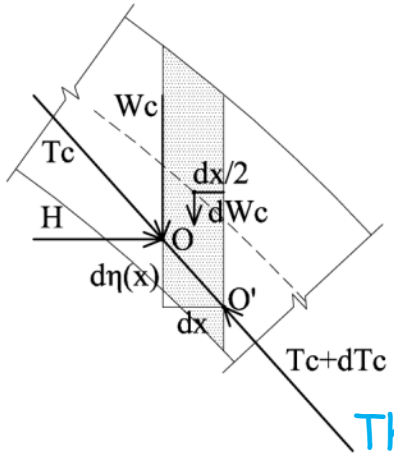
$$-Hdv(x)+W_p dx-dW_p e=0$$

$$\tan\psi_p=\frac{W_p}{H}=\frac{dv(x)}{dx}+\frac{e}{H}\frac{dW_p}{dx}$$

$$\tan\psi_p=\frac{W_p}{H}\neq\frac{dv(x)}{dx}$$

Important conclusion reached by Milankovitch (1904): The direction of the thrust-force ($\tan\psi_p = W_p/H$) is not tangent to the thrust-line





$$-Hd\eta(x)+W_c dx-dW_c\frac{dx}{2}=0$$

$$\tan\psi_c=\frac{W_c}{H}=\frac{d\eta(x)}{dx}+\frac{1}{2}\frac{dW_c}{H}$$

$$\tan\psi_c=\frac{W_c}{H}=\frac{d\eta(x)}{dx}$$

The direction of the thrust-force ($\tan\psi_p = W_p/H$) is tangent to the thrust-line





Médaille d'or, pour cristaux ;

A MM. Chagot frères, au Creusot près Mont-Cénis, Saône-et-Loire, et à Paris, Boulevard Poissonnière, n°. 11 (n°. 1639 *bis*),

Pour cristaux, rappel d'une *Médaille d'or*, qui leur fut décernée en 1819 ;

A la Manufacture royale des glaces à Saint-Gobin, et à Paris (n°. 1347),

Pour glaces coulées des plus grandes dimensions, rappel d'une *Médaille d'or*, qui fut décernée à cet établissement en 1806 ;

A MM. Georgeon et Bontemps, à Choisy-le-Roi (n°. 1282),

Médaille d'argent, pour cristaux, glaces soufflées, verres plats et autres ;

A la Manufacture de glaces de Cirey, Meurthe (n°. 1468),

Pour glaces étamées et non étamées, rappel d'une *Médaille d'argent*, qui lui fut décernée en 1819 ;

A M. de Violaine, à Prémontré, Aisne (n°. 358),

Médaille de bronze, pour verres de toutes dimensions et de toutes couleurs.

1823

MÉMOIRE

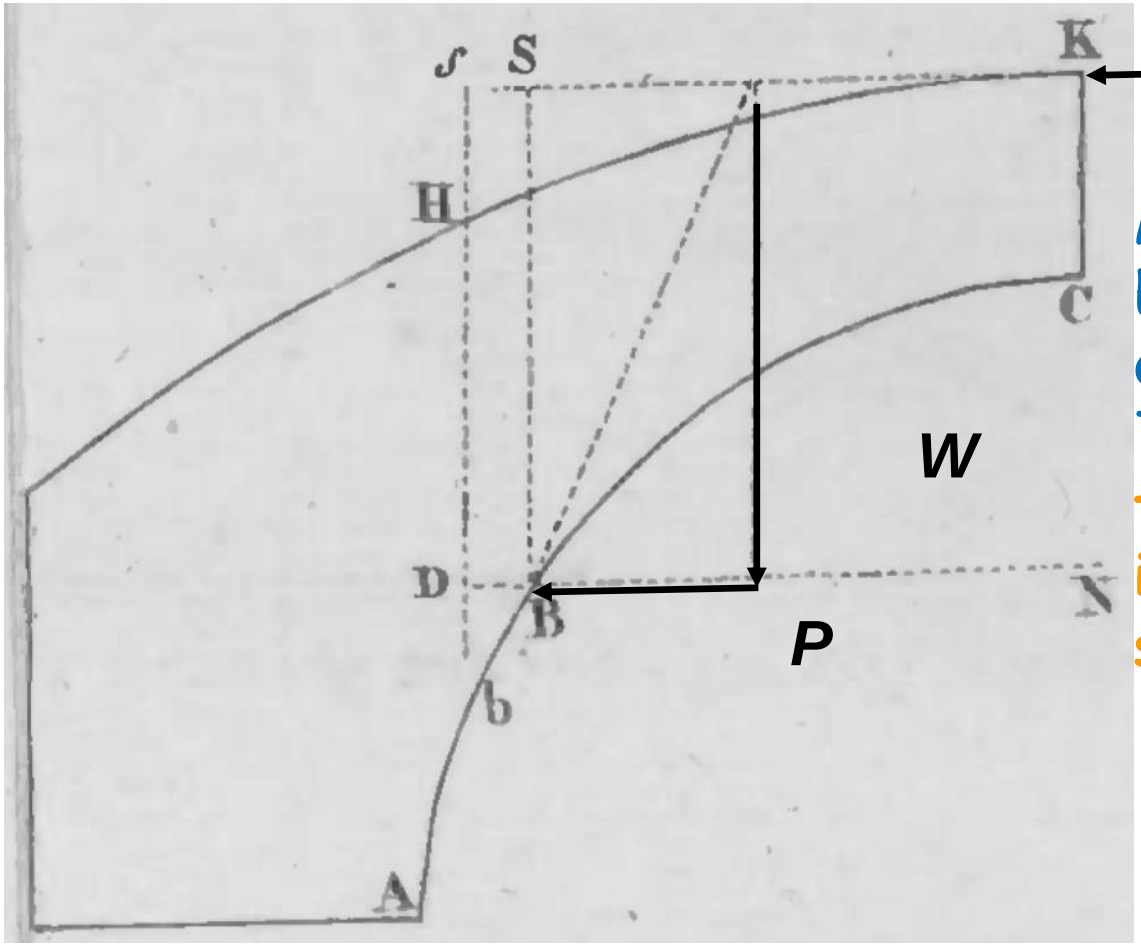
SUR LA STABILITÉ DES VOUTES,

PAR M. G. LAMÉ et E. CLAPEYRON,

Anciens élèves de l'École polytechnique, aspirans-ingénieurs au Corps royal des mines, majors du Génie des voies et communications au service de Russie.

LE projet adopté pour la reconstruction de l'église Saint-Isaac, située sur la place du palais d'hiver à St.-Pétersbourg, présente deux portiques semblables à celui du Panthéon de Rome ; chacun d'eux doit être recouvert par une voûte en berceau et en plein cintre et par deux plates-bandes latérales. La construction de cette voûte, de plus de 40 pieds de diamètre, assise sur des colonnades, sans autre massif latéral pour résister à la poussée, paraît d'une difficulté très-grande, et plusieurs personnes en nient la possibilité. C'est pour examiner si ces doutes ont quelques fondemens, qu'on nous a proposé de calculer la poussée de cette voûte, et d'indiquer les précautions à prendre pour en assurer la stabilité ; on nous a priés, à la même occasion, de reconnaître si le mur cylindrique qui devait soutenir le dôme principal de l'édifice, aurait une épaisseur suffisante pour en empêcher la dégradation.

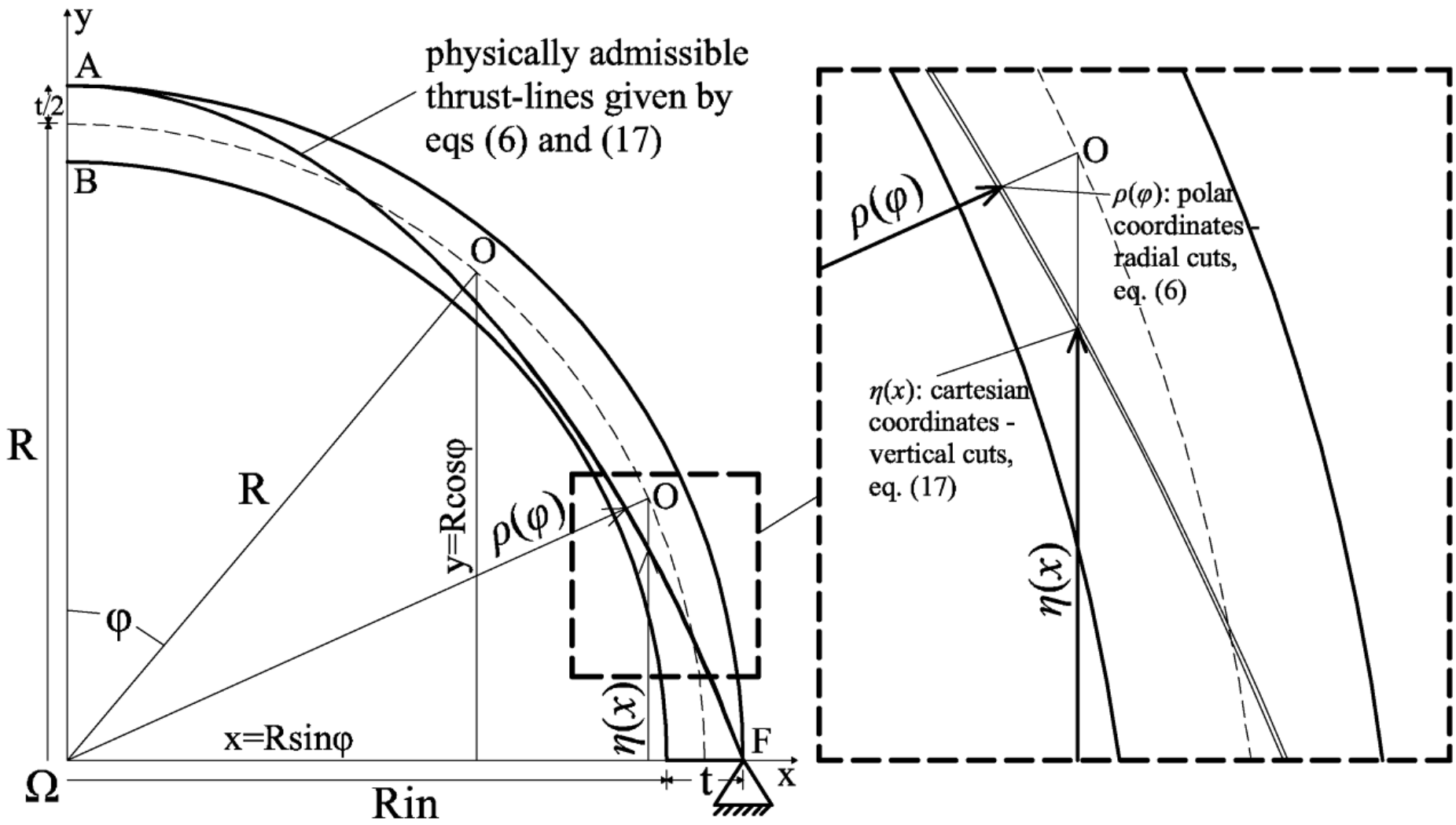
Parmi les hypothèses sur lesquelles on a fondé la théorie de l'équilibre des voûtes, la seule admissible est celle qui le ramène à l'équilibre de quatre leviers pesans, assemblés à charnières, égaux en poids aux deux portions de la voûte



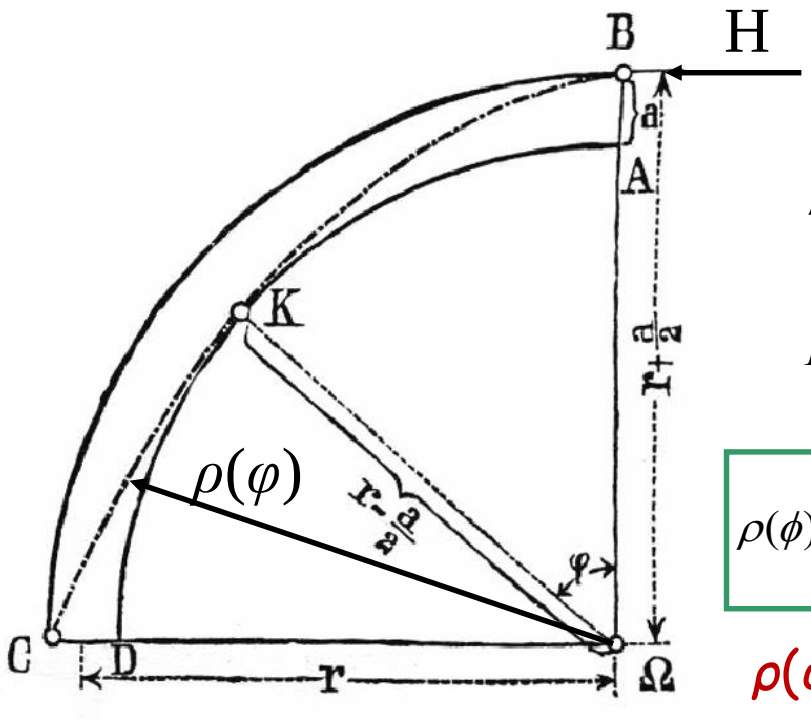
M. G. Lamé & E. Clapeyron's introduction of a vertical cut. This is apparently the **first time when attention is given to the stereotomy exercised.**

M. G. Lamé & E. Clapeyron[(1)] indicated that if one takes vertical cross sections the rupture location would be found from the condition that the tangent to the intrados at point B passes through the intersection of the forces *W* and *P*

(1) Lamé, G. and Clapeyron, E. (1823). Mémoire sur la stabilité des voûtes, *Annales des Mines*, 8.


y
A
t/2
B
physically admissible
thrust-lines given by
eqs (6) and (17)
O
R
R
y=Rcosφ
ρ(φ)
O
φ
x=Rsinφ
η(x)
F
Ω
t
x
Rin
ρ(φ)
O
ρ(φ): polar
coordinates -
radial cuts,
eq. (6)
η(x): cartesian
coordinates -
vertical cuts,
eq. (17)
η(x)

Radial Stereotomy-Polar Coordinate System Geometric Formulation



$$\rho(\phi) = \frac{\rho_o H + \frac{1}{6} t(t^2 + 12R^2)\sin^2\frac{\phi}{2}}{H\cos\phi + tR\phi\sin\phi}$$

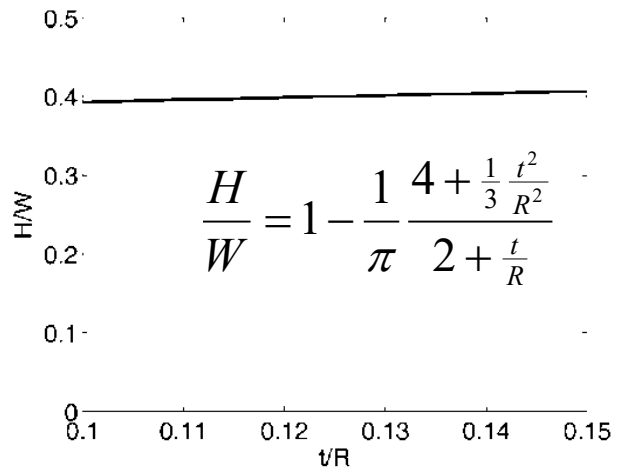
$$\rho(0) = \rho_o = R + \frac{t}{2} \qquad \rho(\pi/2) = R + \frac{t}{2}$$

$$H = tR\,\frac{3\pi(2+\frac{t}{R}) - (12+\frac{t^2}{R^2})}{6(2+\frac{t}{R})}$$

Application of the 2nd BC gives the expression of the minimum force H.

$$\rho(\phi) = R\,\frac{3\pi(2+\frac{t}{R})^2 - (2+\frac{t}{R})(12+\frac{t^2}{R^2})\cos\phi}{6\pi(2+\frac{t}{R})\cos\phi - 2(12+\frac{t^2}{R^2})\cos\phi + 12(2+\frac{t}{R})\phi\sin\phi}$$

ρ(φ) is NOT the equation of a catenary!



$$\frac{H}{W} = 1 - \frac{1}{\pi}\,\frac{4 + \frac{1}{3}\frac{t^2}{R^2}}{2 + \frac{t}{R}}$$

When the arch reaches its minimum thickness ρ(φ) touches the intrados at the imminent intrados hinge. At this location:

dρ(φ)/dφ=0 ρ(φ)=R-t/2

t/R=0.10748, φ=54.484°

published in German in 1904 and later in 1907

ZEITSCHRIFT
FÜR MATHEMATIK UND PHYSIK.

BEGRÜNDET 1856 DURCH † O. SCHLÖMILCH.

FRÜHER HERAUSGEGEBEN VON O. SCHLÖMILCH (1856–1899) UND M. CANTOR (1859–1900).

ORGAN FÜR ANGEWANDTE MATHEMATIK.

GEGENWÄRTIG

UNTER MITWIRKUNG VON C. VON BACH, R. HELMERT, F. KLEIN, C. VON LINDE, H. A. LORENTZ, H. MÜLLER-BRESLAU, H. SEELIGER, H. WEBER

HERAUSGEGEBEN VON

R. MEHMKE IN ANNEPARK UND **C. RUNGE** IN GÖTTINGEN.

55. BAND. 1/2. HEFT.

MIT 58 FIGUREN IM TEXT.

Ausgegeben am 10. Oktober 1907.

LEIPZIG,
DRUCK UND VERLAG VON B. G. TEUBNER.
1907.

Zum Generalregister zu Band 1—50 der Zeitschrift für Mathematik und Physik. Bearbeitet von Professor Dr. E. Wölffing, Stuttgart. [XII u. 308 S.] gr. 8. geh. n. Mk. 15.—, in Leinwand geb. n. Mk. 16.—

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Theorie der Druckkurven.

Von Dr. M. Milankovitch in Wien.

Erstes Kapitel.

Die Druckkurve, ihre Gleichung und ihre Eigenschaften.

Definition der Druckkurve.

1. Es stelle $ABCD$ (Fig. 2, S. 5) einen Teil eines festen Tragkörpers dar, der sich unter dem Einflusse äußerer Kräfte — Lasten und Stützenwiderstände — und des eigenen Gewichtes im Gleichgewichte befindet.

Wir setzen voraus:

Der Tragkörper sei prismatisch, seine Erzeugenden seien senkrecht zur Bildebene, die Leitlinien $CN_1N'B$ und DN_1NA stetige Kurven (Begrenzungskurven), der Tragkörper sei homogen und sein spezifisches Gewicht sei g.

Die Tiefe des betrachteten Teiles des Tragkörpers, — senkrecht zur Bildebene gemessen — den wir zwischen zwei zur Bildebene parallelen Ebenen eingeschlossen denken, sei β.

Von den Lasten setzen wir voraus, daß sie alle in der Bildebene wirken, über die Begrenzungskurven kontinuierlich verteilt sind und sich von Punkt zu Punkt stetig ändern.

Anmerkung 1. Ist die Belastung des Tragkörpers keine stetige, so kann derselbe in solche Teile getrennt werden, für welche die gemachten Voraussetzungen gelten.

Denkt man sich nun einen ebenen, zur Bildebene senkrechten Fugenschnitt NN' durch den Tragkörper geführt, entfernt den rechten Teil des so zerschnittenen Tragkörpers und stellt das zerstörte Gleichgewicht durch die Kraft R wieder her, so nennt man die Kraft R die Druckkraft des Fugenschnittes NN' und ihren Angriffspunkt E an dem geführten Fugenschnitte den Druckmittelpunkt des Fugenschnittes NN'.

Werden nun nach einem bestimmten Gesetze — nehmen wir an senkrecht zu einer gegebenen Kurve der Bildebene — unendlich viele unendlich nahe ebene Fugenschnitte durch den Tragkörper geführt

Die beiden Gleichungen (17) lauten mit Rücksicht auf (16)

$$(18)\quad \frac{\partial\varphi}{\partial\varphi} = \frac{r}{2}(n+2)\frac{N\frac{\partial Z}{\partial\varphi} - Z\frac{\partial N}{\partial\varphi}}{N^2} = 0 \quad \text{oder} \quad N\frac{\partial Z}{\partial\varphi} - Z\frac{\partial N}{\partial\varphi} = 0,$$

$$(19)\quad \varrho = -\frac{r}{2}(n-2) = \frac{r}{2}(n+2)\frac{Z}{N} \quad \text{oder} \quad -(n-2)N = (n+2)Z.$$

Die Gleichungen (18) und (19) können ersetzt werden durch die Gleichungen:

$$(n+2)\frac{\partial Z}{\partial\varphi} + (n-2)\frac{\partial N}{\partial\varphi} = 0,$$

$$(n+2)Z + (n-2)N = 0.$$

Durch Einsetzung der Werte für Z und N aus (14) und (15) gelangt man zu den Gleichungen:

$$(20)\quad 3\pi(n+2)^2 - (2n^3 - 3\pi n^2 + 24n + 12\pi)\cos\varphi + 6(n^2-4)\varphi\sin\varphi = 0,$$

$$(21)\quad 6(n^2-4)\sin\varphi + (2n^3 - 3\pi n^2 + 24n + 12\pi)\sin\varphi + 6(n^2-4)\varphi\cos\varphi = 0.$$

Wird die Gleichung (20) mit $\sin\varphi$, Gleichung (21) mit $\cos\varphi$ multipliziert und werden beide Gleichungen addiert, so folgt:

$$\pi(n+2)\sin\varphi + (n-2)\sin 2\varphi + 2(n-2)\varphi = 0,$$

woraus:

$$(22)\qquad n = 2\frac{2\varphi + \sin 2\varphi - \pi\sin\varphi}{2\varphi + \sin 2\varphi + \pi\sin\varphi}.$$

Aus der Gleichung (21) folgt:

$$(23)\qquad 2n\frac{n^2+12}{n^2-4} - 3(\pi - 2) + 6\varphi\operatorname{cotg}\varphi = 0.$$

Substituiert man den Wert für n aus (22) in die Gleichung (23), so bekommt man eine Gleichung mit der einen Unbekannten φ, woraus man bestimmt:

Dann folgt aus (22)

$$\varphi = 54^\circ 29'.^{1})$$

$$n = \frac{a}{r} = 0{,}1075.$$

Beispiele.

17. Man konstruiere ein unbelastetes Gewölbe, welches den Forderungen genügt: 1. eine seiner Stützlinien falle mit der Gewölbsachse zusammen, 2. die Breite δ der vertikal zu führenden Fugenschnitte sei

1) Ritter gibt an $\varphi = 54^\circ 10'$, Pilgrim gibt an $\varphi = 54^\circ 27'$. Die Werte für n stimmen bei beiden Autoren mit den unserigen überein.

The "hanging chain" = catenary (alysoid)

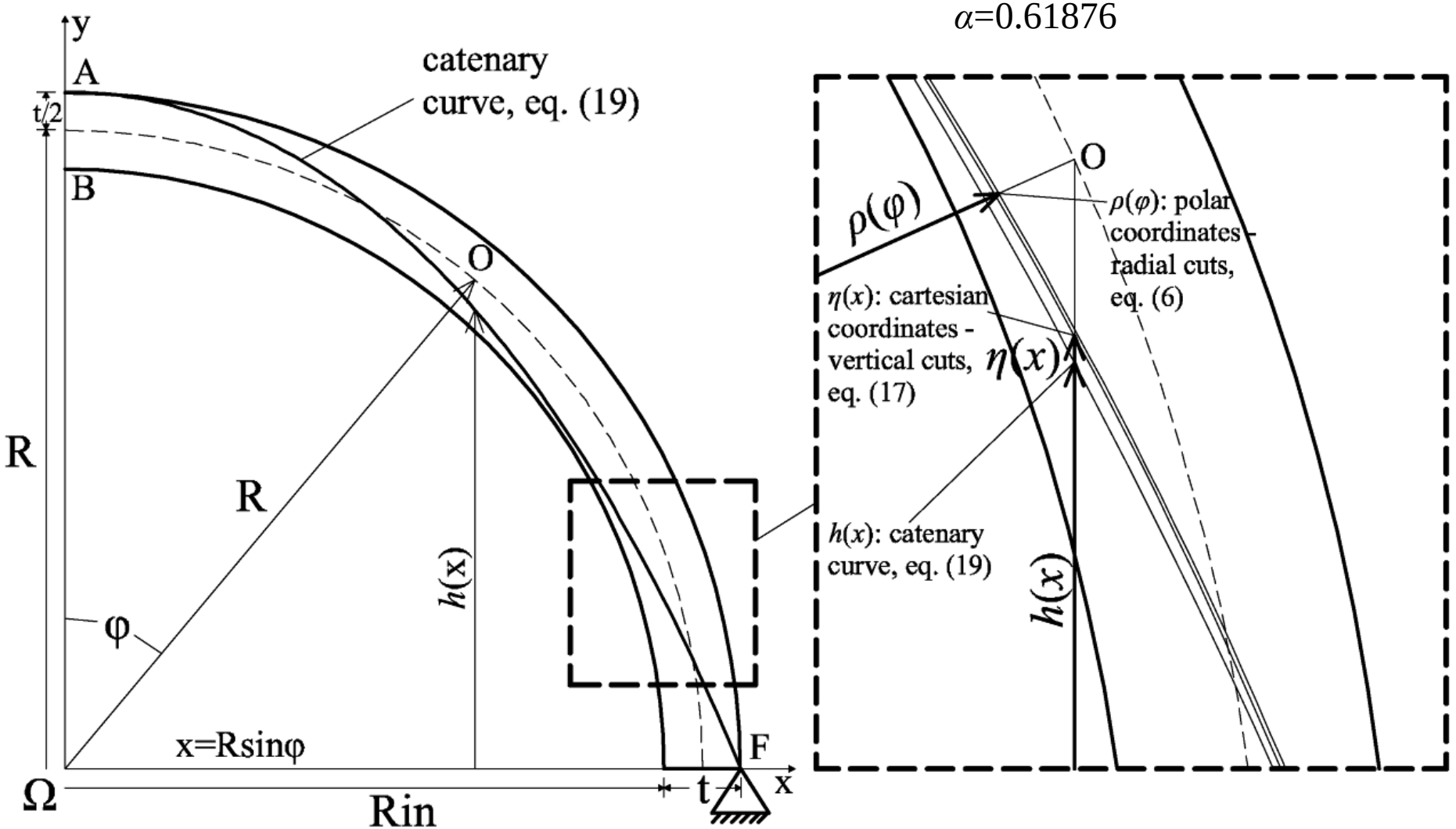
$$h(x) = R_{ex}[1 + a(1 - \cosh(\frac{x}{aR_{ex}}))]$$

$Rex = R + t/2$

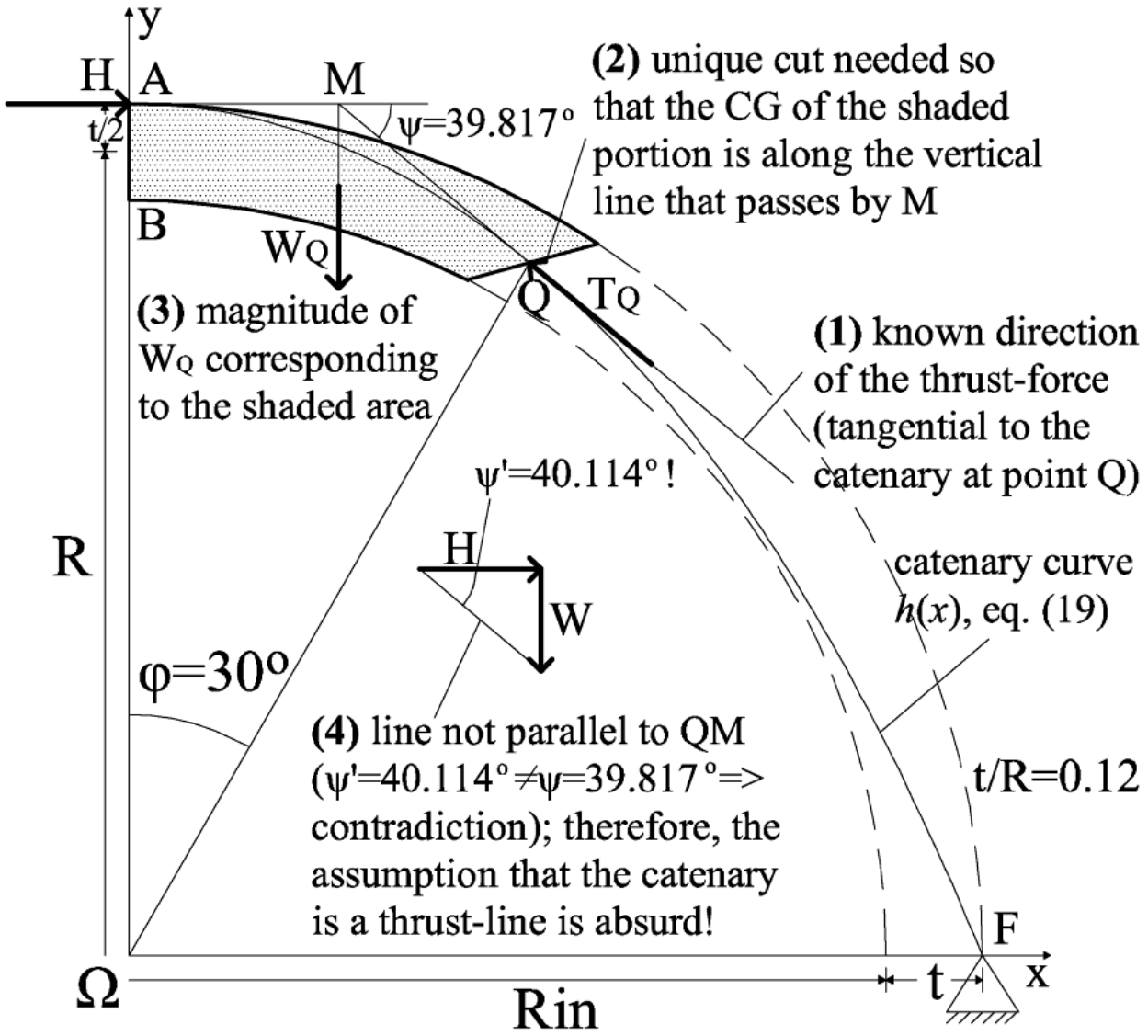
$h(0) = R + t/2 = Rex$

$h(Rex) = 0$

$\alpha = 0.61876$

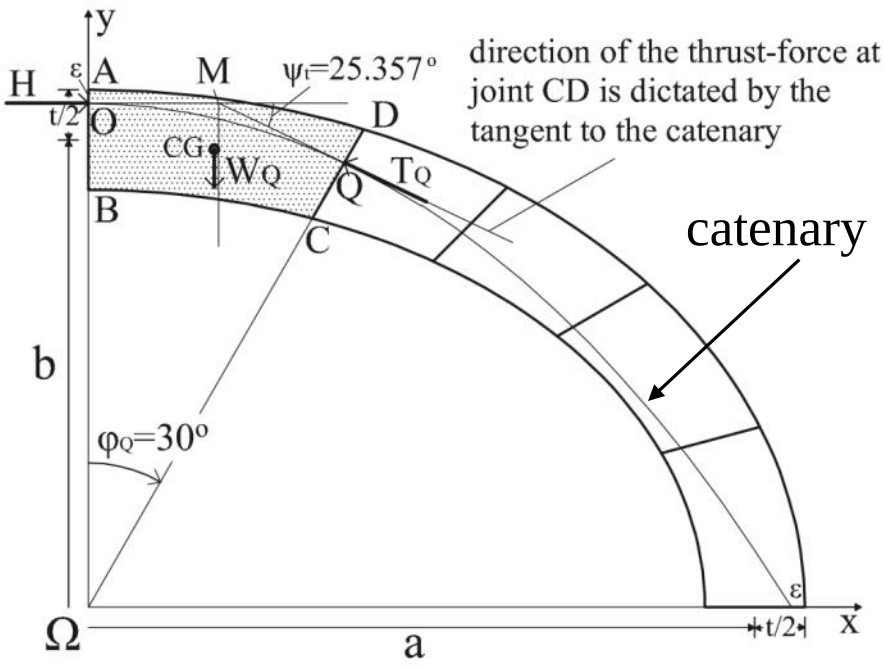


Confirmation that the catenary (the "*hanging chain*") is NOT a physically admissible thrust-line



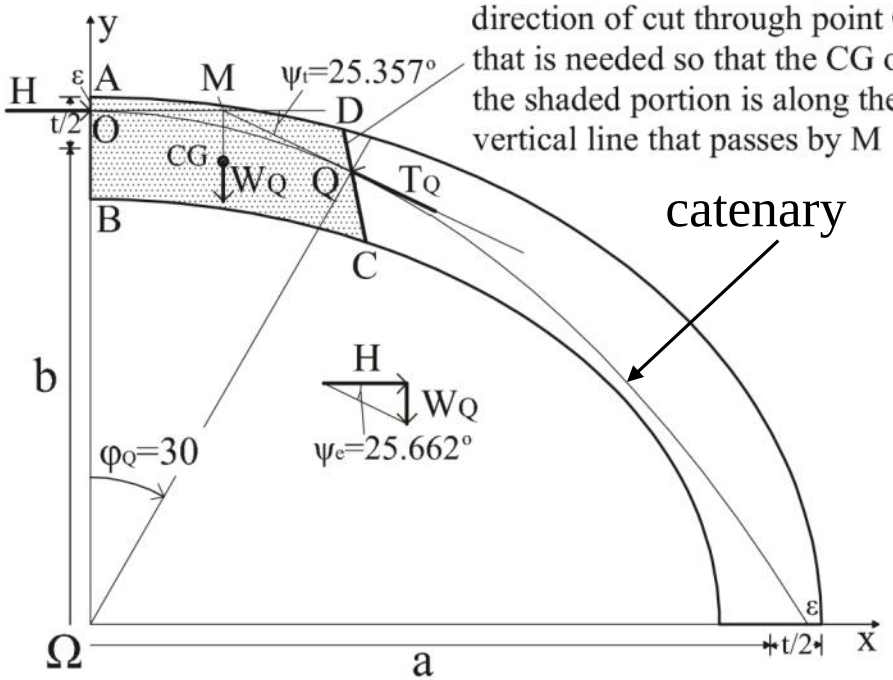
Is the catenary (the "hanging chain") a physically admissible thrust-line of the symmetric masonry arch?

Symmetric masonry arch with physical joints



$x_{CG}/a=0.189 \neq x_M/a=0.195$

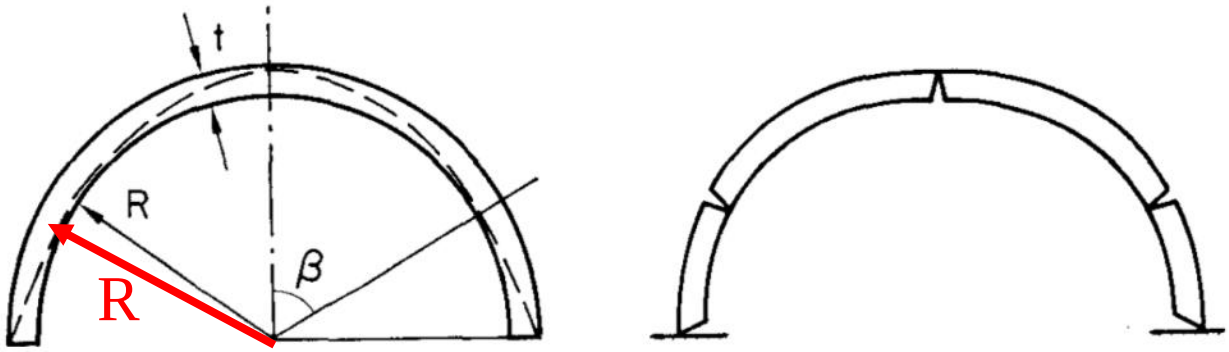
Symmetric arched monolith with zero tensile strength



$\psi_t=25.357° \neq \psi_e=25.662°$

Makris, N., Alexakis, H.: The effect of stereotomy on the shape of the thrust-line and the minimum thickness of semicircular masonry arches. Arch. Appl. Mech. **83**, 1511–1533 (2013). doi:10.1007/s00419-013-0763-4

Jacque Heyman (1969)[1] - Geometric Formulation



(a) $$\beta \cot \beta \left[\frac{2\beta \cos \beta + \sin \beta \cos^2 \beta + \sin \beta}{2\beta \cos \beta + \sin \beta \cos^2 \beta - \sin \beta \cos \beta} \right] = \frac{\pi}{2}.$$

(b) $$\frac{t}{R} = 2 \frac{(\beta - \sin \beta)(1 - \cos \beta)}{\beta(1 + \cos \beta)}.$$

(c) Assumption: The direction of the thrust-force at the intrados hinges lies tangent to the intrados

t/R=0.106, β=φ=58.9°

(1) Heyman, J., **1969.** The safety of masonry arches. Int. J. Mech. Sci. 11, 363-385.

Later Developments

2002: J. Ochsendorf completes his PhD thesis[1] at Cambridge under the supervision of J. Heyman and using a "work-balance" equation verifies Milankovitch's (1904, 1907) result ($t/R=0.10748$, $\varphi=54.484^{o}$) which is referenced in his thesis.

2009: J. Heyman recognizes his unconservative solution[2]; and introduces the idea of multiple solutions depending on the stereotomy exercised

(1) Ochsendorf, J., 2002. Collapse of masonry structures. PhD thesis, Dept. of Engineering, Univ. of Cambridge, Cambridge, U.K.

(2) Heyman, J., 2009. La Coupe des Pierres. Proc. 3rd int. cong. on Construction History, Cottbus, 2, pp 807-812.

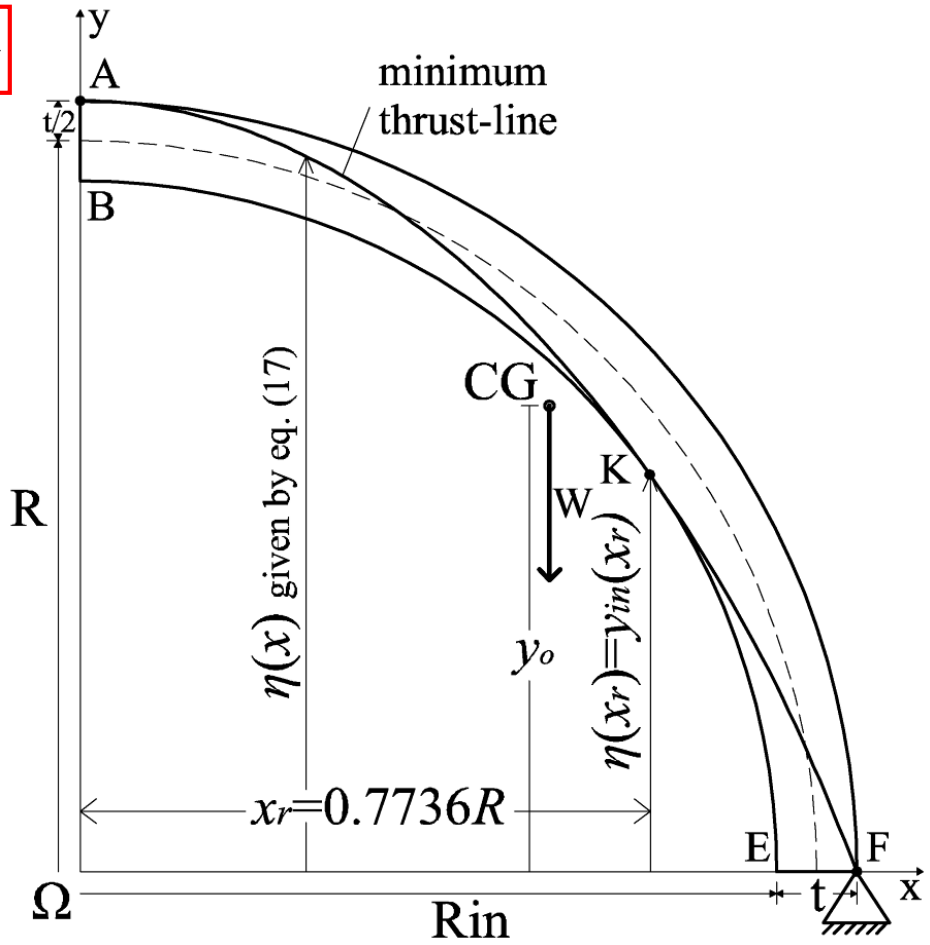
Vertical Stereotomy-Cartesian Coordinate System Geometric Formulation

$$\eta(x) = \eta_o - \int_0^x \frac{W(x)}{H}\,dx$$

$$\tan\psi_c = \frac{W_c}{H} = \frac{d\eta(x)}{dx}$$

$$\eta(x) = R + \frac{t}{2} - \frac{\int_0^x W(x)dx}{W\cdot\left(1-\frac{1}{\pi}\frac{4+\frac{1}{3}\frac{t^2}{R^2}}{2+\frac{t}{R}}\right)}$$

$$c_1 = 1+\frac{1}{2}\frac{t}{R} \qquad c_2 = 1-\frac{1}{2}\frac{t}{R}$$



$$y_{ex}(x) = R\sqrt{\left(1+\frac{1}{2}\frac{t}{R}\right)^2 - \frac{x^2}{R^2}} = R\sqrt{c_1^2 - \frac{x^2}{R^2}}$$

$$y_{in}(x) = R\sqrt{\left(1-\frac{1}{2}\frac{t}{R}\right)^2 - \frac{x^2}{R^2}} = R\sqrt{c_2^2 - \frac{x^2}{R^2}}$$

$$W(x) = W\frac{1}{\pi}\left[\pi + \frac{x}{t}\left(\sqrt{c_1^2 - \frac{x^2}{R^2}} - \sqrt{c_2^2 - \frac{x^2}{R^2}}\right) - \frac{R}{t}\left(c_1^2\arccos\left(\frac{x}{Rc_1}\right) + c_2^2\arccos\left(\frac{x}{Rc_2}\right)\right)\right] \qquad \text{for } 0\le x\le R-t/2$$

$$W(x) = W\frac{1}{\pi}\left(\pi + \frac{x}{t}\sqrt{c_1^2 - \frac{x^2}{R^2}} - \frac{R}{t}c_1^2\arccos\left(\frac{x}{Rc_1}\right)\right) \qquad \text{for } R-t/2\le x\le R+t/2$$

Equation of the minimum thrust-line after adopting a cartesian coordinate system

$$\eta(x) = R + \frac{t}{2} - \frac{y_{ex}(x)(2c_1^2\frac{R}{t} + \frac{x^2}{Rt}) - y_{in}(x)(2c_2^2\frac{R}{t} + \frac{x^2}{Rt}) - 2\frac{R^2}{t}(c_1^3 - c_2^3) + 3x\frac{R}{t}(\pi\frac{t}{R} - c_1^2\arccos(\frac{x}{Rc_1}) + c_2^2\arccos(\frac{x}{Rc_2}))}{3\pi - (12 + \frac{t^2}{R^2})\Big/(2 + \frac{t}{R})}$$

for $0 \le x \le R - t/2$

$$\eta(x) = \eta(R - \frac{t}{2}) - \frac{y_{ex}(x)(2c_1^2\frac{R}{t} + \frac{x^2}{Rt}) - 3R[c_2(\pi - c_1^2\frac{R}{t}\arccos(\frac{c_2}{c_1})) + \sqrt{2}\frac{R}{t}\sqrt{\frac{t}{R}}(c_1^2 - \frac{2}{3}\frac{t}{R})] + 3x\frac{R}{t}(\pi\frac{t}{R} - c_1^2\arccos(\frac{x}{Rc_1}))}{3\pi - (12 + \frac{t^2}{R^2})\Big/(2 + \frac{t}{R})}$$

for $R - t/2 \le x \le R + t/2$

$\eta(x)$ in NOT the equation of a catenary!

$$\eta(x_r) = y_{in}(x_r) \qquad \frac{d\eta(x)}{dx}\Big|_{x=x_r} = \frac{dy_{in}(x)}{dx}\Big|_{x=x_r}$$

Makris, N., Alexakis, H.: The effect of stereotomy on the shape of the thrust-line and the minimum thickness of semicircular masonry arches. Arch. Appl. Mech. **83**, 1511–1533 (2013). doi:10.1007/s00419-013-0763-4

t/R=0.10946, φ=54.923°

result from cartesian coordinates that serves safety

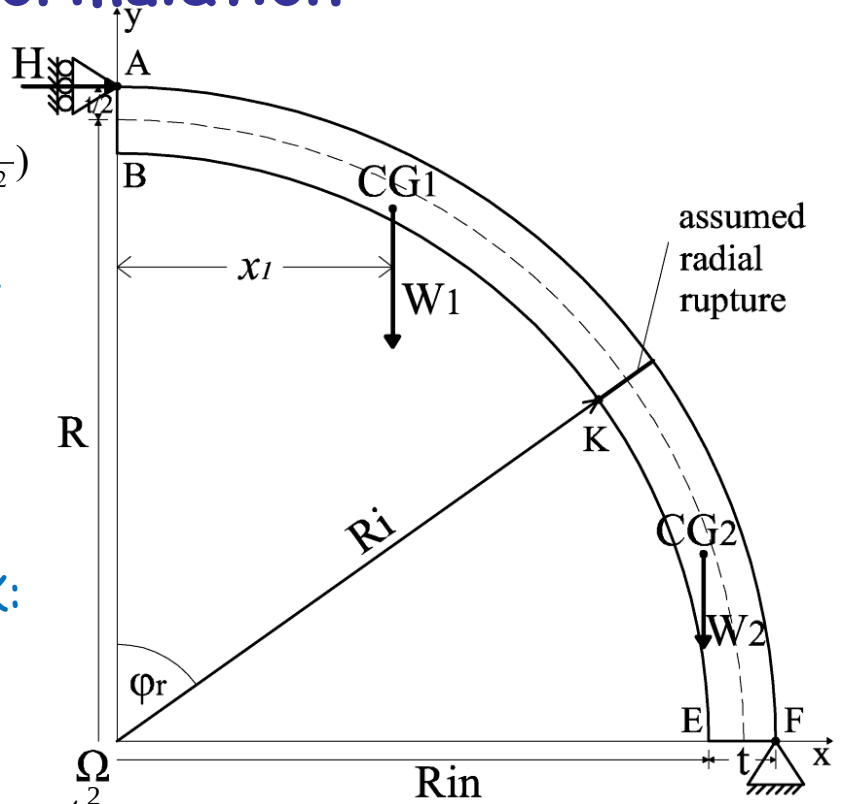
t/R=0.10748, φ=54.484°

result from polar coordinates, Milankovitch (1904,1907)

Radial Stereotomy-Polar Coordinate System Variational Formulation

$$W_1 = tR\phi_r \qquad W_2 = tR(\tfrac{\pi}{2} - \phi_r)$$

$$x_1 = R\frac{1-\cos\phi_r}{\phi_r}(1+\frac{1}{12}\frac{t^2}{R^2}) \qquad x_2 = R\cdot 2\frac{\cos\phi_r}{\pi - 2\phi_r}(1+\frac{1}{12}\frac{t^2}{R^2})$$



Moment equilibrium of the half arch about point F:

$$H = tR\frac{3\pi(2+\frac{t}{R}) - (12+\frac{t^2}{R^2})}{6(2+\frac{t}{R})}$$

Moment equilibrium of part 1 about point K:

$$H[R+\frac{t}{2} - (R-\frac{t}{2})\cos\phi_r] = W_1[(R-\frac{t}{2})\sin\phi_r - x_1]$$

$$[2\frac{t}{R}(\frac{t^2}{R^2}+12) - 3\pi(\frac{t^2}{R^2}-4)]\cos\phi_r = 3\pi(\frac{t}{R}+2)^2 + 6(\frac{t^2}{R^2}-4)\phi_r\sin\phi_r$$

$$a_1(\phi_r)(\frac{t}{R})^3 + a_2(\phi_r)(\frac{t}{R})^2 + a_3(\phi_r)\frac{t}{R} + a_4(\phi_r) = 0$$

For a given value of φ_r one obtains 3 roots for t/R

$$a_1(\phi_r) = 2\cos(\phi_r) \qquad a_2(\phi_r) = -3[\pi(1+\cos(\phi_r)) + 2\phi_r\sin(\phi_r)]$$

$$a_3(\phi_r) = -12(\pi - 2\cos(\phi_r)) \qquad a_4(\phi_r) = -12[\pi(1-\cos(\phi_r)) - 2\phi_r\sin(\phi_r)]$$

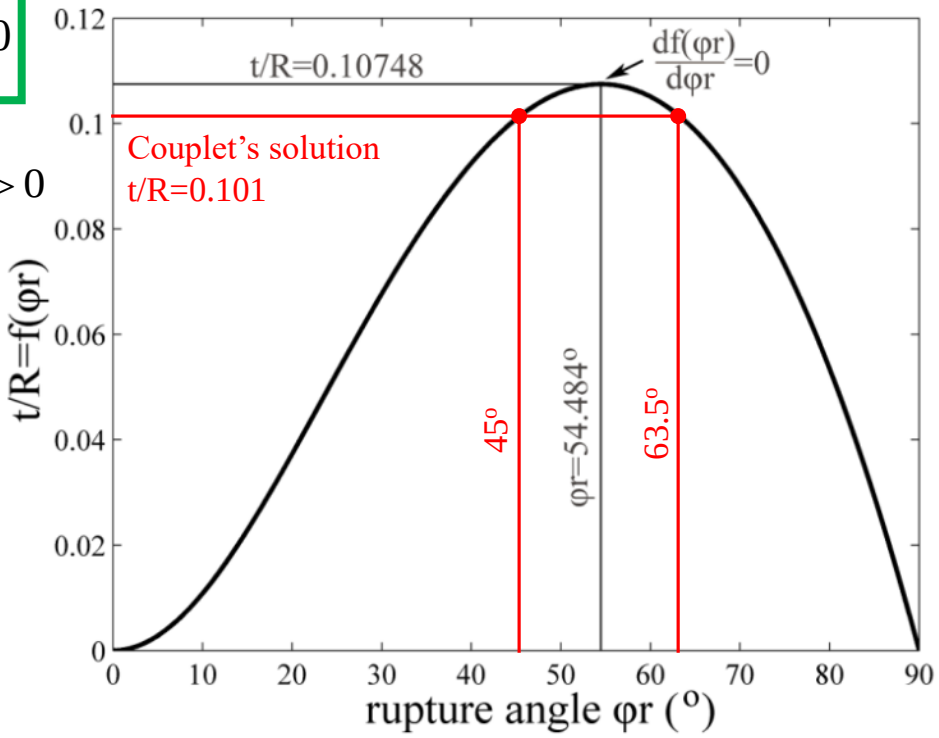
$$a_1(\phi_r)(\frac{t}{R})^3 + a_2(\phi_r)(\frac{t}{R})^2 + a_3(\phi_r)\frac{t}{R} + a_4(\phi_r) = 0$$

$$\Delta = 18\alpha_1\alpha_2\alpha_3\alpha_4 - 4\alpha_2^3\alpha_4 + \alpha_2^2\alpha_3^2 - 4\alpha_1\alpha_3^3 - 27\alpha_1^2\alpha_4^2 > 0$$

$\frac{t}{R}|_1 < 0$ Rejected

$\frac{t}{R}|_2 = f(\phi_r)$ Accepted

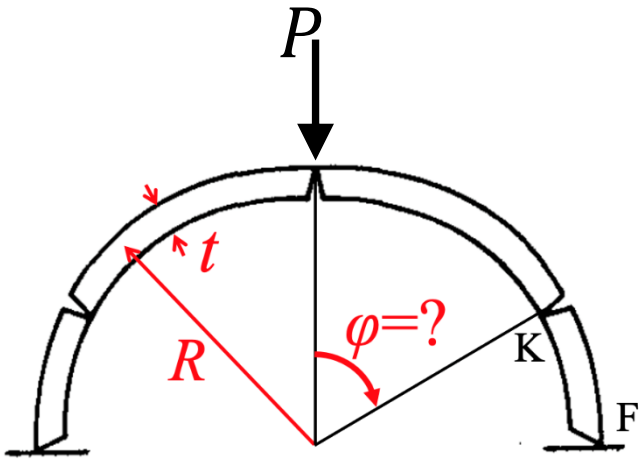
$\frac{t}{R}|_3 > 10$ Rejected



A variational statement of equilibrium

$$\underbrace{\delta\Pi}_{\text{Variational indicator}} = \underbrace{P\delta v}_{\text{External Work}} - \underbrace{\delta V}_{\text{Stored potential or … strain energy}} = 0$$

Principle of Stationary Potential Energy



Principle of Stationary Potential Energy

Potential energy of the semicircular arch:

$$V = W \cdot y_o = R^3 \frac{t}{R}\left(1 + \frac{1}{12}\frac{t^2}{R^2}\right)$$

where $y_o = R(12+(t/R)^2)/6\pi$, $\quad W = \frac{\pi}{2} tR$

Since $t/R = f(\varphi_r)$:

$$V(\phi_r) = R^3 f(\phi_r) \cdot \left(1 + \frac{1}{12} f(\phi_r)^2\right)$$

The masonry arch is a typical structural system where **the only forces (weights) acting in the system are conservative** and where **the work of all forces is accounted by the potential energy $V(\varphi_r)$.** According to the principle of stationary potential energy, **the geometrically admissible hinged mechanism** is in an equilibrium state if and only if the **total potential energy is stationary**:

$$\delta V(\phi_r) = \frac{dV(\phi_r)}{d\phi_r}\delta\phi_r = 0$$

$$\delta V(\phi_r) = \frac{dV(\phi_r)}{d\phi_r}\delta\phi_r = 0 \quad \Rightarrow \quad \frac{d}{d\phi_r}[\, f(\phi_r)\cdot(1+\frac{1}{12} f(\phi_r)^2)] = 0$$

or:

$$\frac{df(\phi_r)}{d\phi_r}[1+\frac{1}{4} f(\phi_r)^2)] = 0$$

The quantity in brackets is always positive so $\delta V(\phi_r) = 0$ only if:

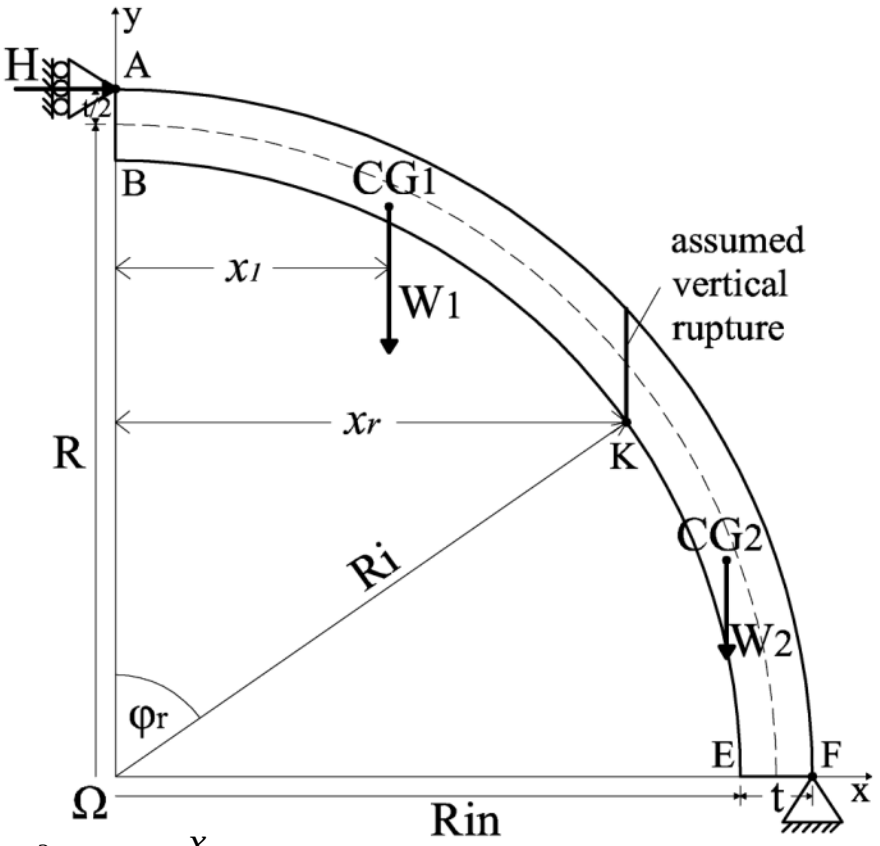
$$\frac{df(\phi_r)}{d\phi_r} = \frac{d(t/R)}{d\phi_r} = 0$$

Makris, N., Alexakis, H.: The effect of stereotomy on the shape of the thrust-line and the minimum thickness of semicircular masonry arches. Arch. Appl. Mech. **83**, 1511–1533 (2013). doi:10.1007/s00419-013-0763-4

which is satisfied when: $t/R=0.10748,\ \varphi=54.484^o$

This is precisely the result obtained by Milankovitch (1904), derived now with a variational formulation.

Vertical Stereotomy-Cartesian Coordinate System Variational Formulation



Moment equilibrium of part 1 about point K:

$$H\left(R+\frac{t}{2}-y_{in}(x_r)\right)=W_1(x_r-x_1)$$

where:

$$H=tR\frac{3\pi(2+\frac{t}{R})-(12+\frac{t^2}{R^2})}{6(2+\frac{t}{R})}$$

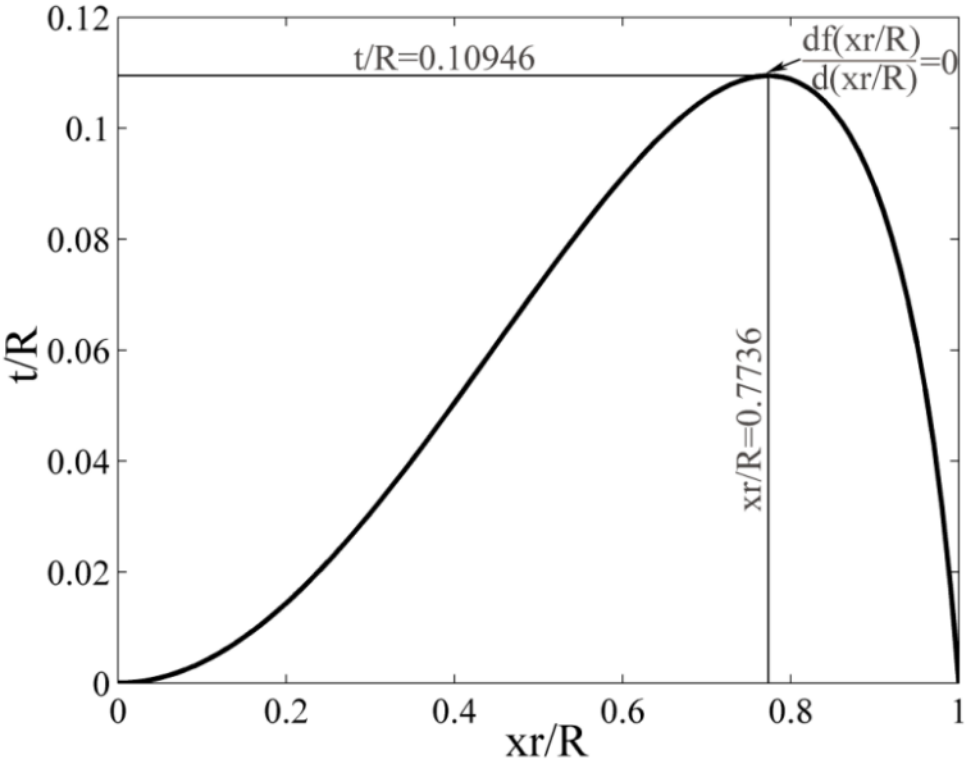
$$W_1=R^2\frac{1}{2}\left[\frac{x_r}{R}\left(\sqrt{c_1^{\,2}-\frac{x_r^{\,2}}{R^2}}-\sqrt{c_2^{\,2}-\frac{x_r^{\,2}}{R^2}}\right)+c_1^{\,2}\arcsin\left(\frac{x_r}{Rc_1}\right)-c_2^{\,2}\arcsin\left(\frac{x_r}{Rc_2}\right)\right]$$

$$x_1=R\frac{2\frac{t}{R}\left(1+\frac{1}{12}\frac{t^2}{R^2}\right)-\frac{2}{3}\left[\left(c_1^{\,2}-\frac{x_r^{\,2}}{R^2}\right)^{3/2}-\left(c_2^{\,2}-\frac{x_r^{\,2}}{R^2}\right)^{3/2}\right]}{\frac{x_r}{R}\left(\sqrt{c_1^{\,2}-\frac{x_r^{\,2}}{R^2}}-\sqrt{c_2^{\,2}-\frac{x_r^{\,2}}{R^2}}\right)+c_1^{\,2}\arcsin\left(\frac{x_r}{Rc_1}\right)-c_2^{\,2}\arcsin\left(\frac{x_r}{Rc_2}\right)}$$

Numerical solution of moment equation:

$$H(R+\frac{t}{2}-y_{in}(x_r))=W_1(x_r-x_1)$$

gives the graph:



where $\frac{t}{R}=f(\frac{x_r}{R})$

The potential energy of the arch is:

$$V(\frac{x_r}{R})=R^3 f(\frac{x_r}{R})\cdot(1+\frac{1}{12}f(\frac{x_r}{R})^2)$$

The geometrically admissible hinged mechanism with a vertical rupture is in equilibrium state if and only if the potential energy is stationary: $\delta V(\frac{x_r}{R})=0$

$$\frac{df(\frac{x_r}{R})}{d\frac{x_r}{R}}=\frac{d(\frac{t}{R})}{d\frac{x_r}{R}}=0$$

$t/R=0.10946,\ \varphi=54.923^o$

This result obtained with a variational formulation is precisely the same result obtained earlier with a geometric formulation.

Minimum allowable thickness and rupture locations of a semicircular monolith with zero tensile strength

Radial Cuts **Polar Coordinate System**		**Vertical Cuts** **Cartesian Coordinate System**		Minimum thickness of a semicircular arch needed to accommodate a catenary curve
Rupture angle	Minimum thickness ($R=R_{in}+t/2$)	Rupture angle	Minimum thickness ($R=R_{in}+t/2$)	
φ_r=BΩK=54.484º	**t/R=0.10748**	**φ_r=BΩK=54.923º**	**t/R=0.10946**	**t/R=0.11166≈1/9=0.11111**
•M. Milankovitch (1904, 1907): Geometric Solution •J. Ochsendorf (2002): Trial-and-error solution of the "work balance" equation •This work: Principle of Stationary Potential Energy		•This work: (a) Geometric Solution (b) Principle of Stationary Potential Energy		•This work: Geometric Solution The catenary curve is not a physically admissible thrust-line; therefore the point that touches the intrados of the arch (φ_N=56.746º) is not an imminent hinge.

Makris, N. and H. Alexakis, "The Effect of Stereotomy on the Shape of the Thrust-Line and the Minimum Thickness of Semicircular Masonry Arches", *Archive of Applied Mechanics,* Vol. 83, Issue 10, 2013, pp. 1511-1533.

ral, on doit considérer la limite XZ comme trop éloignée du point de rupture. D'ailleurs ce qu'on doit se proposer dans la pratique est d'augmenter la stabilité de la voûte en employant le moins possible de matériaux : or il est aisé de voir qu'une masse donnée augmente d'autant plus le moment de stabilité, qu'elle se trouve placée plus près de la verticale passant par le point de rupture : c'est donc près de cette ligne sur-tout qu'il est convenable de charger la voûte.

3. Le point de rupture correspond au cas où $\frac{ma}{h}$ est un *maximum*, ainsi que nous venons de l'établir : or, pour qu'on puisse calculer ce *maximum*, et par conséquent déterminer la position du point de rupture, il faut connaître suivant quelle ligne se fait la séparation de la voûte ; l'expérience apprend qu'elle se dirige d'abord normalement à la courbe avec le plan de joint d'un voussoir, puis se redresse et s'élève jusqu'à l'*extrados*, en suivant une ligne plus ou moins sinueuse. Ainsi la ligne de séparation de la voûte est toujours plus ou moins à gauche de la verticale passant par le point de rupture. Pour fixer cette incertitude et être sûr de ne pas arriver à des résultats trop faibles dans la pratique, nous supposerons que la fente se fasse dans la direction à laquelle correspond la valeur *maximum* du moment de BCKH par rapport à BS (*fig.* 2) ; c'est-à-dire suivant la verticale B S elle-même, car toute portion de la masse tendant à se détacher, qui serait située à gauche de B S, diminuerait le moment total en donnant un terme négatif.

Dans cette hypothèse, ma représente le moment de la surface BCKH, BH étant vertical, par rapport à BS. Or, le point de rupture, devant correspondre au cas où $\frac{ma}{h}$ est un *maximum*, est celui pour lequel cette expression est constante lorsqu'on passe d'un point B au point infiniment voisin b ; il faut pour cela que l'on ait : ma est à h comme la différentielle du moment ma est à celle de h. Or, en passant du point de B au point b infiniment voisin sur la courbe *intrados*, h augmente de bD ; et ma de $m \times$ BD, plus le moment de la masse additionnelle BbsS, par rapport à la verticale bs, ce dernier moment est un infiniment petit du second ordre, et disparaît devant $m \times$ BD : on a donc au point de rupture.....

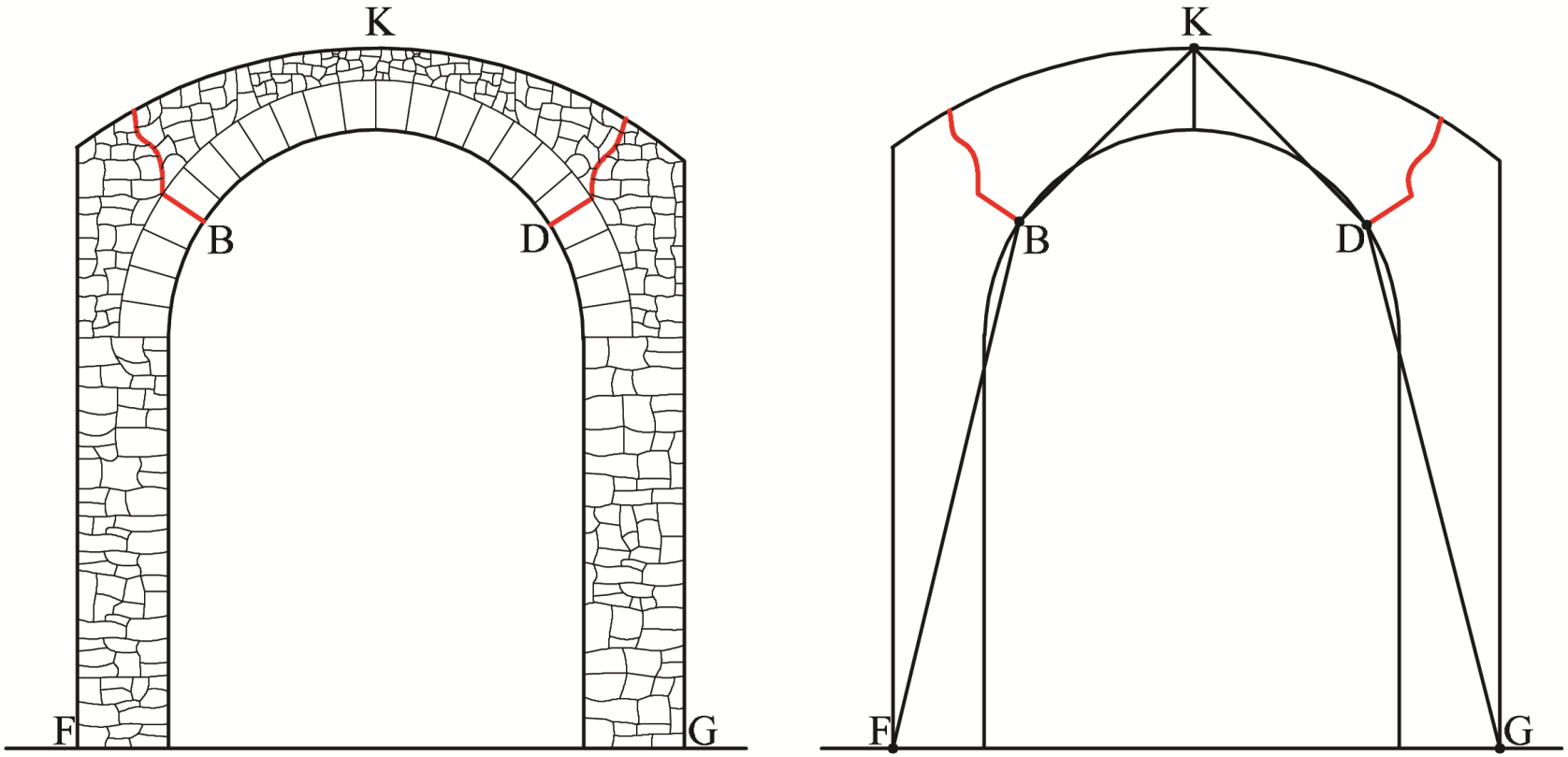
$$\frac{ma}{h} = \frac{m \times \mathrm{BD}}{b\mathrm{D}} \text{ ou } \frac{a}{h} = \frac{\mathrm{BD}}{b\mathrm{D}}.$$

Ce qui apprend *que le point de rupture est celui pour lequel la tangente à l'*intrados, *en ce point, vient couper l'horizontale passant par le sommet de la clef, au même point que la verticale passant par le centre de gravité de la masse tendant à se détacher.* $\frac{ma}{h}$ *est alors égal à la masse m divisée par la tangente de l'angle à l'horizon de la touchante à l'*intrados *au point de rupture.*

Ce résultat remarquable par sa simplicité nous a paru utile à énoncer, parce que, sans exiger

M.G. Lamé and E. Clapeyron: Mémoire sur la stabilité des voûtes (1823)

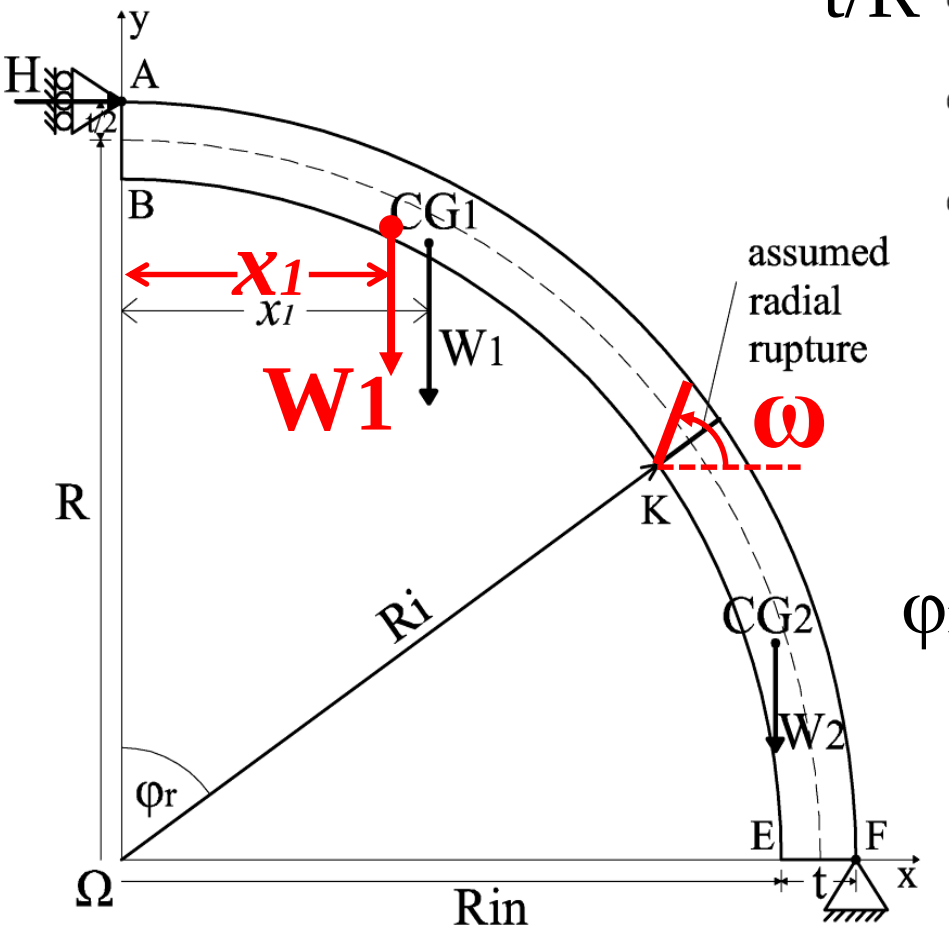
"*l'expérience apprend qu'elle se dirige d'abord normalement à la courbe avec le plan de joint d'un voussoir, puis se redresse et s'élève jusqu'à l'extrados*"

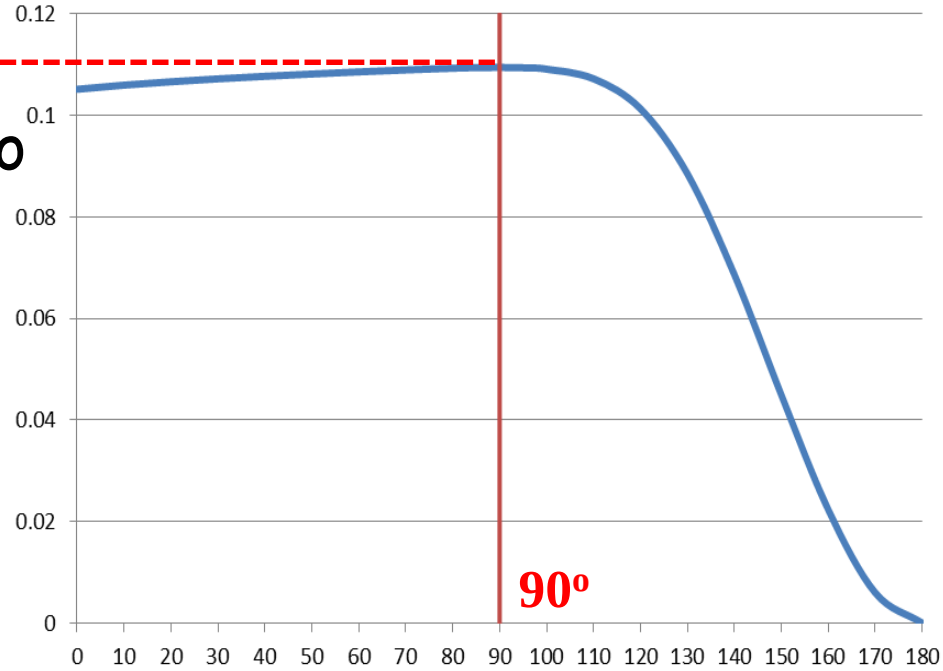


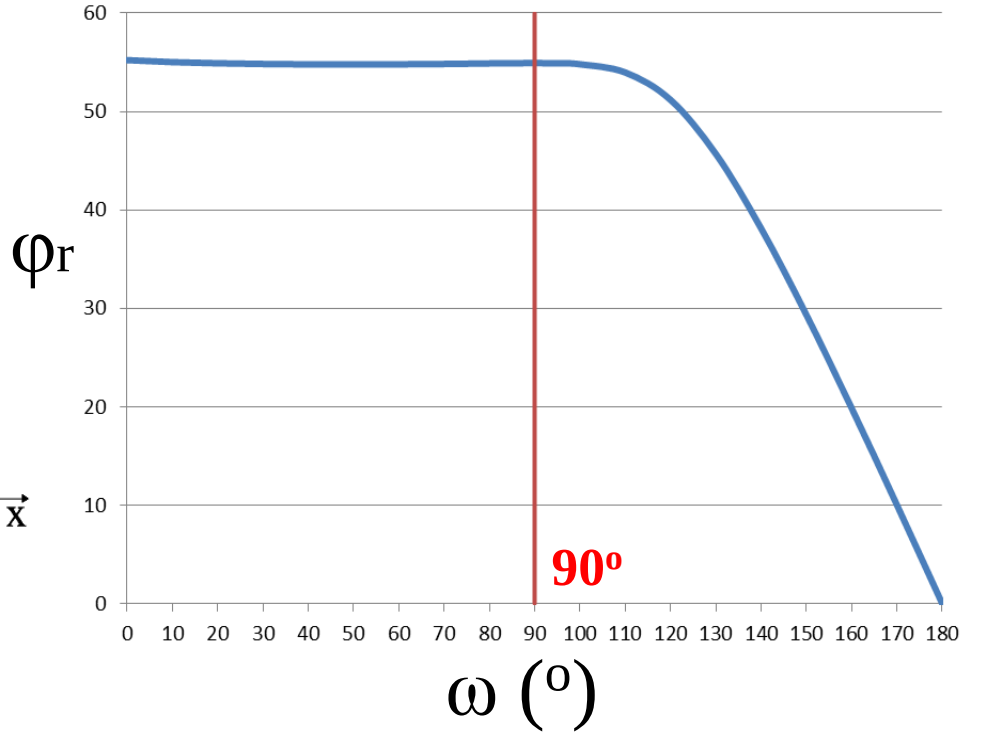
Observed path of a rupture that initiates radially (points B and D) along the physical joint between two adjacent vitruvius voussoirs and subsequently it propagates upwards following a nearly vertical direction.

max: t/R=0.10946

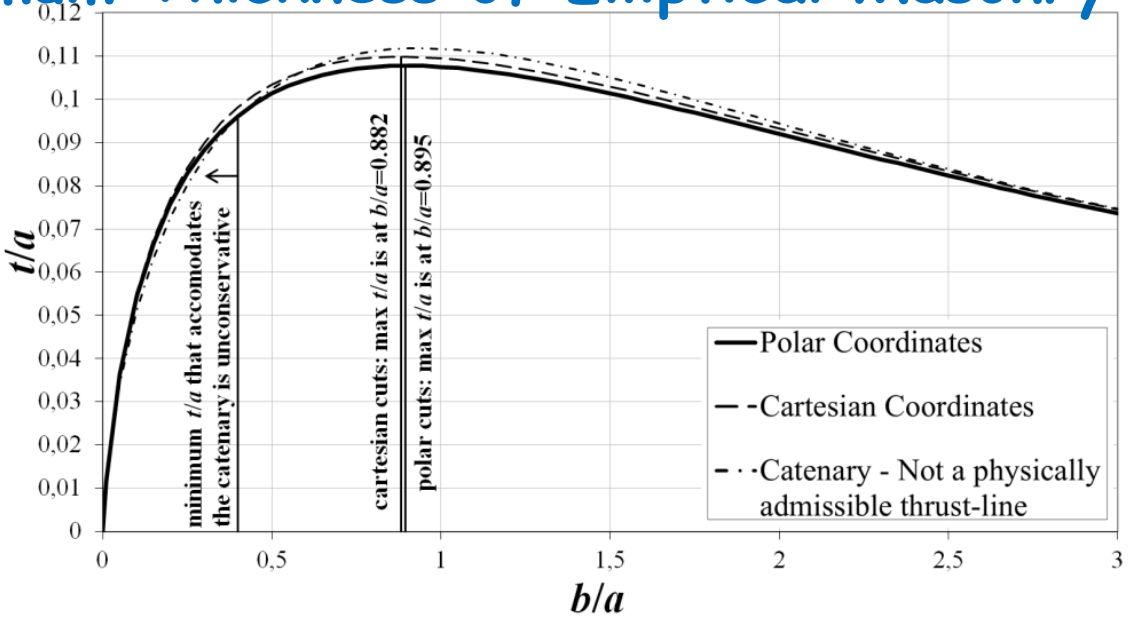
Vertical ruptures corresponds to the lower collapse mechanism

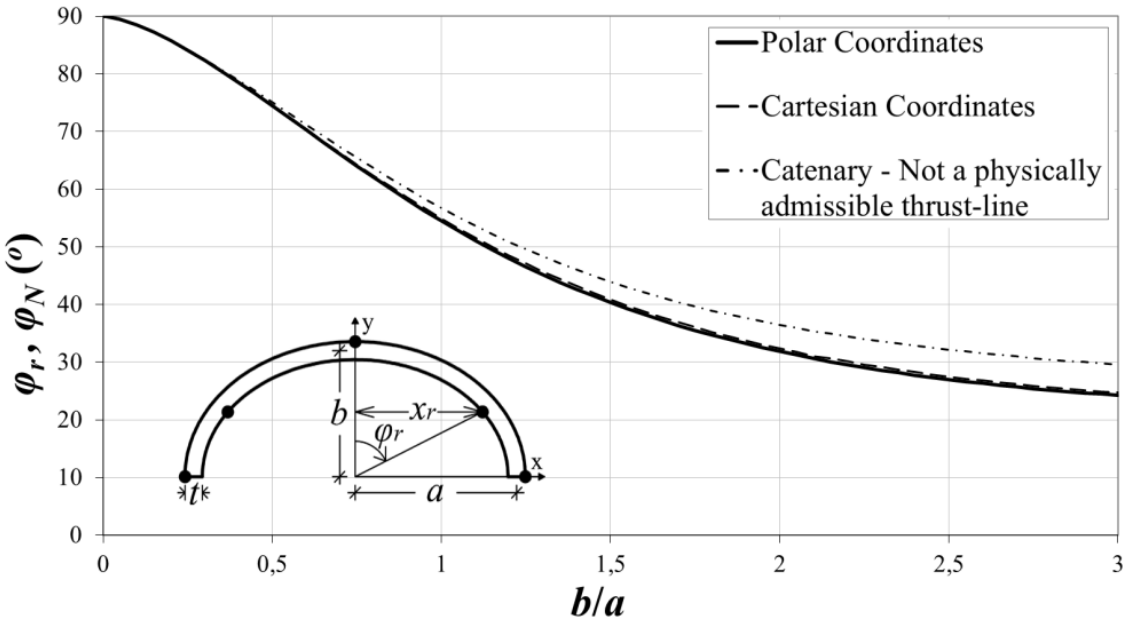


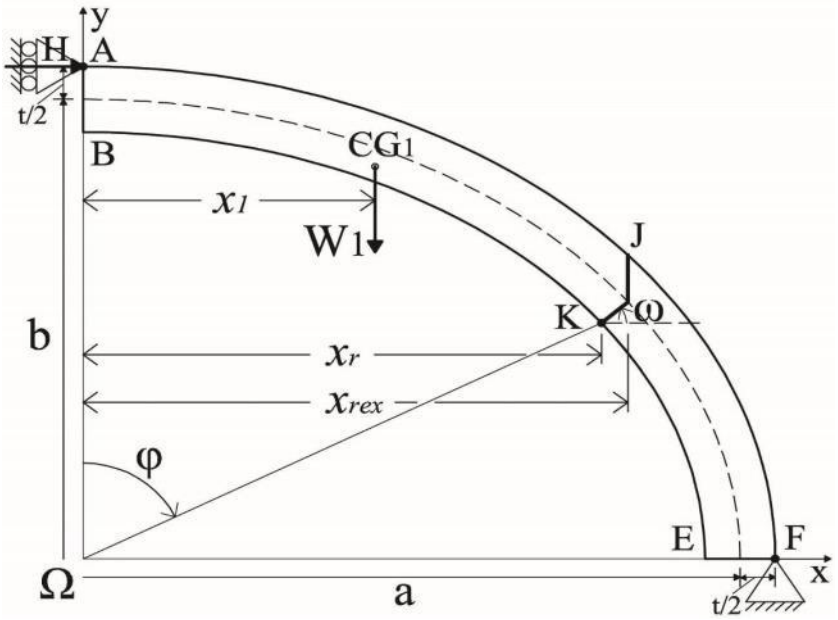




Application of the Variational Method to Calculate the Minimum Thickness of Elliptical Masonry Arches







Formation of a hinge mechanism in an elliptical arch by assuming that the rupture that initiates at the intrados hinge, K, develops along a broken line which initiates perpendicular to the intrados up to the mid-thickness of the arch and subsequently redirects upwards propagating along the vertical.

Minimum thickness *t*/*a* and radial locations φ_r=BΩK of the intrados hinge K of elliptical arches with *b*/*a*=0.5, *b*/*a*=1 (semicircle) and *b*/*a*=1.5 for the various stereotomies examined in this work.

Rupture direction	**b/a=0.5**			**b/a=1**			**b/a=1.5**		
	ω (°)	*t*/*a*	φ_r (°)	ω (°)	*t*/*a*	φ_r (°)	ω (°)	*t*/*a*	φ_r (°)
Radial	15.558	0.10127	74.442				49.677	0.10135	40.323
Perpendicular to intrados	51.315	0.10278	74.395	**35.516**	**0.10748**	**54.484**	26.737	0.10038	40.430
Lamé & Clapeyron's observation	51.193	**0.10309**	74.465	35.291	0.10846	54.709	26.541	0.10163	40.659
Vertical	90.000	0.10340	74.533	90.000	0.10946	54.923	90.000	0.10291	40.875

The educational and practical value of the "*hanging chain*"

The minimum thickness of a circular arch that is needed to accommodate a catenary curve given by: $h(x) = R_{ex}[1 + a(1 - \cosh(\frac{x}{aR_{ex}}))]$

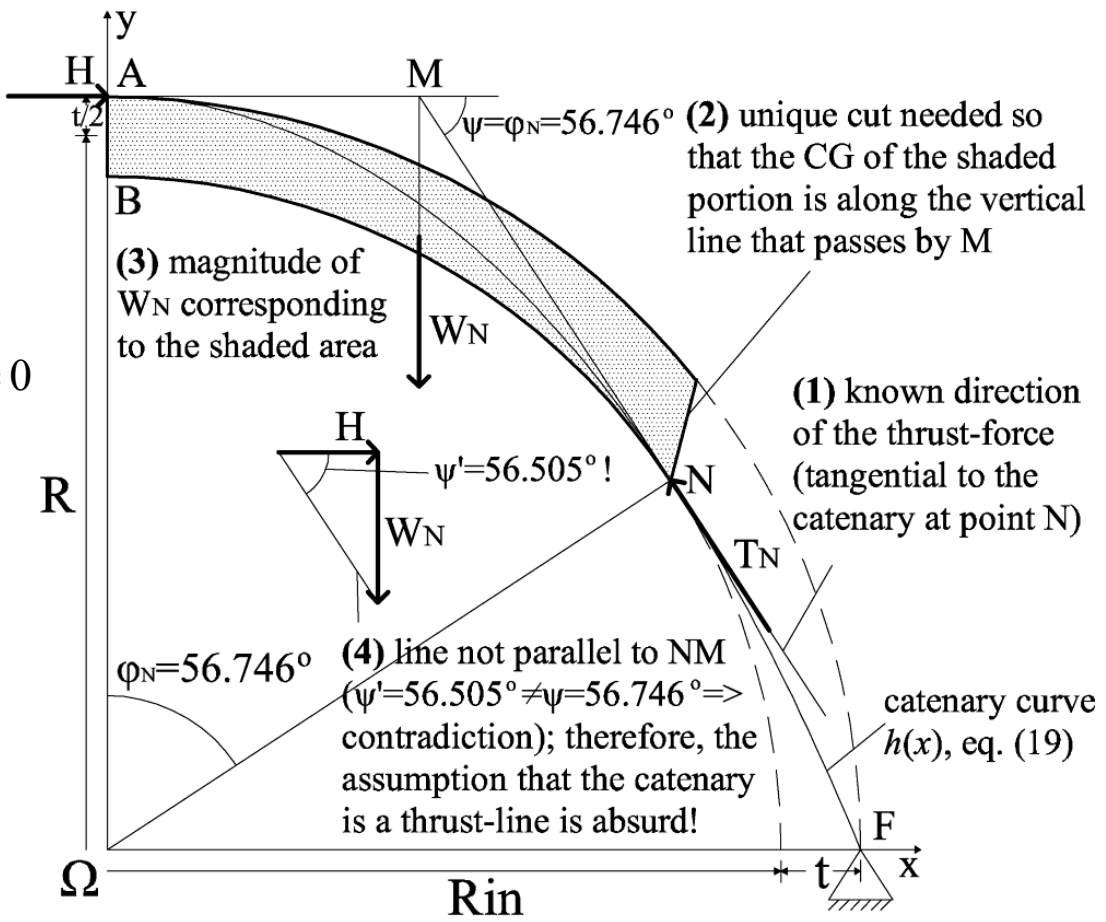
$$h(x_N) = y_{in}(x_N)$$

$$\frac{dh(x)}{dx}\Big|_{x=x_N} = \frac{dy_{in}(x)}{dx}\Big|_{x=x_N}$$

$$c_1[1 + a(1 - \cosh(\frac{1}{ac_1}\frac{x_N}{R}))] - \sqrt{c_2^2 - \frac{x_N^2}{R^2}} = 0$$

$$\sinh(\frac{1}{ac_1}\frac{x_N}{R})\sqrt{c_2^2 - \frac{x_N^2}{R^2}} - \frac{x_N}{R} = 0$$

t/R=0.11166, φ=56.746°



	Radial Cuts Polar Coordinates			Vertical Cuts Cartesian Coordinates			Minimum thickness of an elliptical arch needed to accommodate a catenary (alysoid). The catenary is not a physically admissible thrust-line; therefore, the location x_N (φ_N) where the catenary touches the intrados is not an imminent rupture location.		
b*/*a	Minimum thickness: ***t*/*a***	Rupture angle: φ_r(°)= BΩK	Rupture location: x_r/a	Minimum thickness: ***t*/*a***	Rupture angle: φ_r(°)= BΩK	Rupture location: x_r/a	***t*/*a***	φ_N(°)	x_N/a
0.25	0.08273	84.123	0.86659	0.08422	84.101	0.86487	0.08029	84.213	0.87085
0.5	0.10127	74.442	0.81830	0.10340	74.533	0.81841	0.10242	75.067	0.82647
0.75	0.10709	64.015	0.78906	0.10925	64.299	0.79108	0.11045	65.443	0.80248
1	**0.10748**	**54.484**	**0.77021**	**0.10946**	**54.923**	**0.77359**	**0.11166**	**56.746**	**0.78956**
1.25	0.10513	46.567	0.75828	0.10689	47.090	0.76257	0.10928	49.570	0.78400
1.5	0.10135	40.323	0.75154	0.10291	40.875	0.75642	0.10502	43.951	0.78387
1.75	0.09684	35.516	0.74903	0.09821	36.067	0.75431	0.09986	39.670	0.78789
2	0.09200	31.857	0.75016	0.09324	32.395	0.75568	0.09439	36.446	0.79497

Alexakis, H. and N. Makris, "Minimum Thickness of Elliptical Arches", *Acta Mechanica*, Vol. 224, Issue 12, 2013, pp. 2977-2991.

Conclusions

Pierre Couplet, 1729:	$t/R = 0.101$, $\varphi=45^o$
Joaquín Monasterio, 1805-1810 :	$0.1053 < t/R < 0.1176$
Milutin Milancovitch, 1904-1907:	$t/R = 0.1075$, $\varphi=54.48^o$
Jacque Heyman, 1969	$t/R=0.106$, $\beta=\varphi=58.9^o$
John Ochsendorf, 2002	$t/R=0.1075$, $\varphi=54.48^o$
Makris and Alexakis 2013	$t/R=0.1075$, $\varphi=54.48^o$
Infinite many neighboring solutions	•
depending on the stereotomy exercised	•
Our result serves safety	$t/R=0.1095$, $\varphi=54.92^o$

Why this work has value?

In my mind, during our research as engineers and scientists **every now and then we reach these great moments** when **our understanding transitions from confusion to clarity!**

In November 2024, **Nicos Makris** was elected **foreign member of** the ***Serbian Academy of Sciences and Arts (SASA)*** for showing: ***(a)*** ***that there is an infinite number of neighboring minimum thickness values for the masonry arch, depending on the stereotomy exercised*** *(Milankovitch result is one of them for radial stereotomy); and* ***(b)*** *that the hanging chain (the catenary)* ***is NOT a physically admissible thrust line*** *for the masonry arch.*



Proposed Building Structures for Extraterrestrial Exploration







Merci !

Thank you !