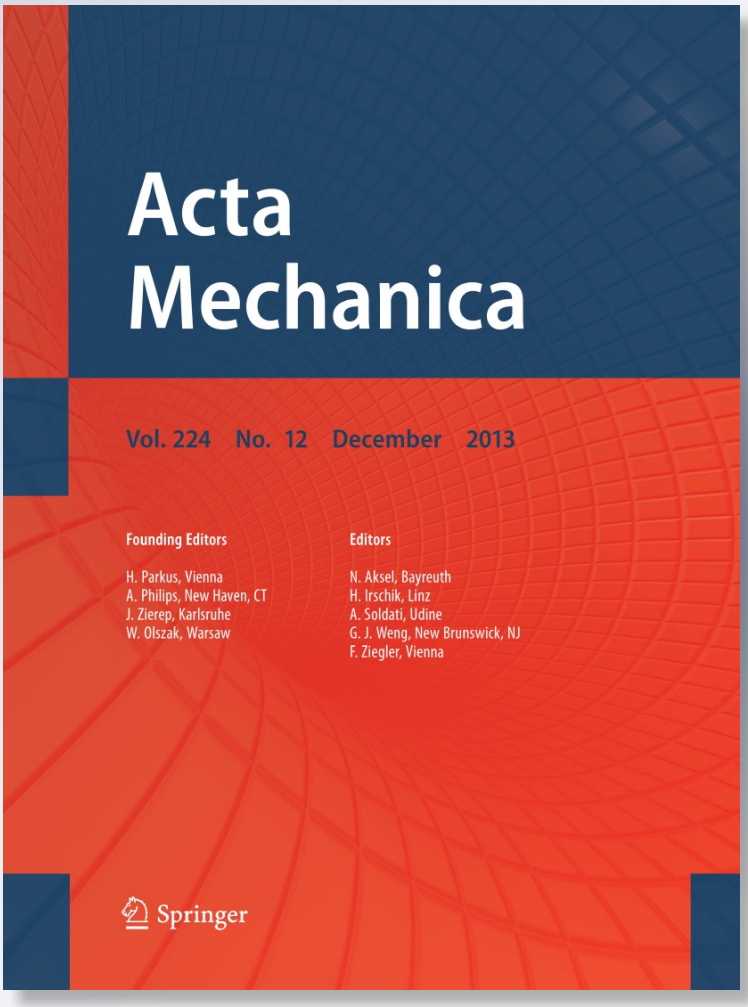
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Abstract In this paper, we compute the minimum thickness and the location of the imminent intrados hinge of symmetric elliptical masonry arches when subjected to their weight. While this problem (Couplet's problem) was solved rigorously for semicircular arches more than a century ago, no results have been available for elliptical arches. Motivated from the recent growing interest in identifying the limit equilibrium states of historic structures, this paper first computes two neighboring physically admissible thrust-lines which can just be located in elliptical arches by adopting either a polar or a cartesian coordinate system. These two distinguishable, physically admissible thrust-lines are neighboring thrust-lines to Hooke's catenary which is not a physically admissible thrust-line as has been shown recently. Accordingly, the paper shows that the answer for the minimum thickness of symmetric elliptical masonry arches is not unique and that it depends on the coordinate system adopted and the associated stereotomy exercised. This result is confirmed by developing a variational formulation after selecting the appropriate directions of the rupture that initiates at the intrados hinge. The paper concludes that Hooke's limiting catenary, although not a physically admissible thrust-line, offers a conservative value for the minimum thickness in most practical configurations.

1 Introduction

The problem of determining the minimum thickness of masonry arches has been a constant challenge to the engineering community through the centuries. While the art of building masonry arches was practiced sparsely during the classical period in Greece and later advanced appreciably during the Roman period with the extensive construction of bridges and aqueducts, it was apparently Hooke [1] who was the first to propose a rational rule for sizing masonry arches by describing the analogy in the load path between a hanging chain which forms a catenary in tension under its own weight and a masonry arch which stands under compression. This analogy conceived by Hooke is expressed in the literature "As hangs the flexible line, so but inverted will stand the rigid arch" (Heyman [2,3], O'Dwyer [4], Block et al. [5] among others). Hooke's statement [1] was published in the form of an anagram and it was not until after his death in 1703 that the anagram was decoded. In the meantime, Gregory [6] had applied the newly invented calculus in an attempt to determine the shape of the hanging chain (see also Heyman [2,7]).

The "hanging chain" analogy proposed by Hooke [1] has been used with remarkable success by several eminent engineers. Poleni [8] implemented Hooke's concept to assess the safety of the cracked dome of St. Peter's Church in Rome and showed that the dome was safe. For this, he divided the dome in slices and hung a

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number of unequal weights proportional to the weight of the corresponding arch sections and then showed that the "hanging chain" could fit within the profile of the arch. Barlow [9] conducted experiments to demonstrate the equivalence between the statics of the hanging chain and the stability of the masonry arch. The "hanging chain" approach led to the concept of the funicular polygon where the shape that a weightless string assumes under a set of concentrated loads when inverted illustrates the load path of compressive forces that develop within an arched structure in order to support the same set of loads. For instance, as late as in the beginning of the twentieth century, the eminent architect A. Gaudí (1852–1926) hung individual weights from hanging chains to design complex structures such as the Church of Colonia Güell near Barcelona, Spain (Roca et al. [10]).

According to R. Hooke's analogy, a masonry arch with zero tensile strength is expected to satisfy equilibrium if its finite thickness can accommodate the shape of an inverted "hanging chain". When no additional concentrated weights are appended, the shape of the catenary (the alysoid) is a physically realizable load path which keeps the constituents of the one-dimensional chain (rings) in equilibrium. When adopting R. Hooke's catenary curve as a candidate thrust-line (line of resistance) in a two-dimensional masonry arch, there is also a silent, a priori commitment on the direction of the thrust-force acting at each joint; since by definition, the thrust-force along a catenary has to be tangent to the catenary curve. This tangency commitment associated with the catenary line of resistance (which has been inherent from the statics of the one-dimensional "hanging-chain") challenges the equilibrium of masonry arches with finite thickness (Makris and Alexakis [11,12], Alexakis [13]); therefore; the catenary curve is not a physically admissible thrust-line, although it is a neighboring line to the physically admissible thrust-lines that one obtains after adopting either a polar or a cartesian coordinate system. Given that the exact profile of the thrust-line has a central role in determining the minimum thickness of masonry arches, this paper investigates in detail the physical admissible thrust-lines of elliptical arches.

The problem of determining the minimum thickness of semicircular masonry arches has challenged the engineering community since the early eighteenth century (Couplet [14]), was tackled with remarkable ingenuity by Monasterio in the early nineteen century (Foce [15], Albuerne and Huerta [16]), was addressed rigorously in the early twentieth century in the nearly unknown work by Milankovitch [17,18], was revisited in the late 1960s by Heyman [7] in association with the stability of historic bridges, and remains worth discussing until today (Ochsendorf [19], Heyman [20]).

The main goal of this paper is to build upon the above-mentioned work on circular arches and compute the minimum thickness of elliptical arches which can support their own weight. This work has been mainly motivated from a recent growing interest to understand and identify the limit equilibrium state of historic structures. Accordingly, this paper first computes the physically admissible thrust-lines which can just be located in elliptical masonry arches by adopting (a) a polar coordinate system and (b) a cartesian coordinate system, and subsequently computes the minimum allowable thickness of elliptical masonry arches by employing (a) a variational formulation and (b) a geometric formulation. The paper shows that the adoption of a cartesian coordinate system yields values for the minimum thickness, which are slightly higher than those computed with a polar coordinate system. The accuracy of these values is confirmed with two independent formulations (variational and geometric) presented together with a numerical solution of the equilibrium analysis of any section of the arch. The paper explains that the "counterintuitive" result of having two different theoretically correct answers for the minimum thickness of masonry arches depending on the coordinate system adopted derives from the "ambitious" request to express the load path in a two-dimensional arch with finite thickness with the thrust-line—that is a concept inherent to a one-dimensional structure (the "hanging chain"). The paper shows that the catenary curve which can just be located in elliptical arches offers a conservative value of the minimum thickness, except for the case of shallow arches ($b/a < 0.4$), where the opposite happens.

2 Physically admissible thrust-lines of an arched monolith with zero tensile strength

The remarkable directness of Hooke's "hanging chain" as a design tool for sizing stable masonry arches is probably the reason that even in the recent literature, it is widely believed that the catenary (the alysoid) is a physically admissible thrust-line of the masonry arch.

The thrust-line (or the line of resistance, Moseley [21], or the *Druckkurve*, Milankovitch [17,18]) is defined as the geometrical locus of the application points of the resultant thrust-force that develops at any cross-section of the arch. Given that the calculation of the minimum thickness of the masonry arch derives from the limiting arch that is thick enough so that it can just accommodate a physically admissible thrust-line, the identification of physical admissible thrust-lines is central in this study.

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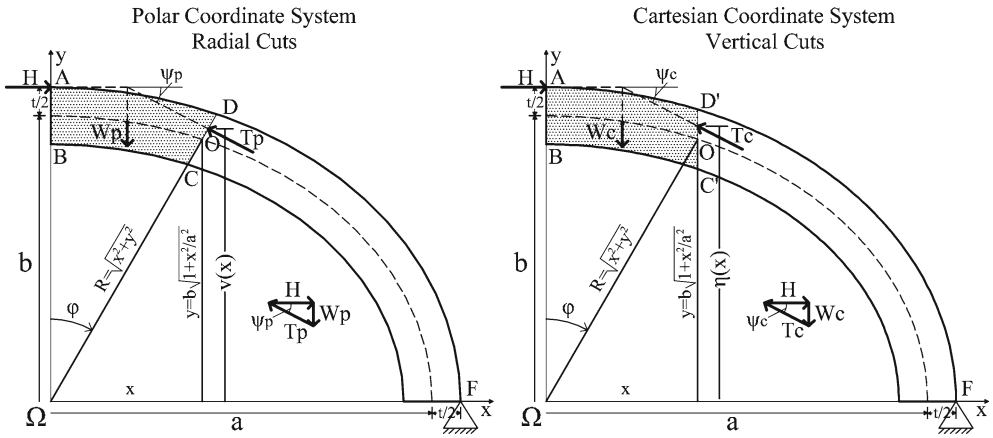


Fig. 1 Monolithic elliptical arch with zero tensile strength, $b/a = 0.75$ and $t/a = 0.15$. At any point O along the *center line* of the *arch* (*dashed line*) with polar coordinates $R = \sqrt{x^2 + y^2}$, $\varphi = \tan^{-1} x/y$, or cartesian coordinates x, $y = b\sqrt{1 - x^2/a^2}$, the thrust-force T_p along a *radial cut* after adopting a polar coordinate system (*left*) is different than the thrust-force T_c along a *vertical cut* after adopting a cartesian coordinate system (*right*). Accordingly, one obtains two distinguishable physically admissible minimum thrust-lines (see Fig. 2) depending on the coordinate system that is selected

The first proof known to the authors that the catenary (alysoid) is not an admissible thrust-line for circular arches is that offered in the work of Milankovitch [17,18] who upon adopting a polar coordinate system derived the following closed form expression for the thrust-line of semicircular arches:

$$\rho(\varphi) = \frac{\rho_o H + \frac{1}{6} t\,(t^2 + 12R^2)\sin^2\frac{\varphi}{2}}{H\cos\varphi + tR\varphi\sin\varphi}. \tag{1}$$

In Eq. (1), H is the horizontal thrust-force at the crown of the symmetric semicircular arch, ρ_o is the distance from the center of the arch to the application point of the horizontal thrust H, and φ is the angular coordinate that starts measuring from the vertical axis of the symmetry. It is worth noting that Milankovitch [17,18] obtains Eq. (1) with two different approaches: (a) via differential calculus and (b) via macroscopic equilibrium of a finite portion of the arch (see also Foce [22]). Clearly, Eq. (1) is not the equation of a catenary.

Consider now the monolithic elliptical arch with semispan a and height b, as shown in Fig. 1, which has zero tensile strength and therefore acts only in compression. Any point O that is located along the mid-thickness line of the arch (dashed line) has coordinates x, $y = b\sqrt{1 - x^2/a^2}$.

The minimum allowable thickness of elliptical arches that is the subject of this study is the thickness that will bring the arch at the verge of becoming a five-hinge mechanism (Couplet [14]); therefore, we are interested in computing thrust-lines that pass by the imminent hinges of the arch at the extrados points A at the crown and F at the springing—as if the arch is supported by a hinge at point F as shown in Fig. 1. The term "minimum" thrust-line is used in this work for any physically admissible thrust-line that passes by points A and F since the associated horizontal thrust-force at the crown (point A) is a minimum. The original idea of the minimum (or maximum) thrust-force was first advanced by Coulomb [23] in his general theory on the stability of arches, a work that was fundamental in all the subsequent research on this subject. With the points A and F being established, we are interested in computing the resultant thrust-force acting within the arch at location O with coordinates x, y. The standard way to proceed is to make a cut passing through point O and work the force and moment equilibrium of the top portion of the arch down to the cut. One way to cut the elliptical arch is along the radial direction after adopting the polar coordinate system $R = \sqrt{x^2 + y^2}$, $\varphi = \tan^{-1} x/y$ as shown in Fig. 1(left). With this choice, we isolate the portion of the arch ABCOD. Another way to cut is along the vertical direction after adopting a cartesian coordinate system x, $y = b\sqrt{1 - x^2/a^2}$ as shown in Fig. 1 (right). With this choice, we isolate the portion of the arch ABC′OD′, which has different size and weight than the portion ABCOD; therefore, the thrust T_c along the vertical cut C′OD′ is different both in magnitude and in direction than the thrust T_p along the radial cut COD. As this exercise is repeated for different points O along the mid-thickness line of the arch, one obtains two neighboring, yet different thrust-lines—one thrust-line that results from successive vertical cuts after adopting a cartesian coordinate system and a different thrust-line that results from successive radial cuts after adopting a polar coordinate system.

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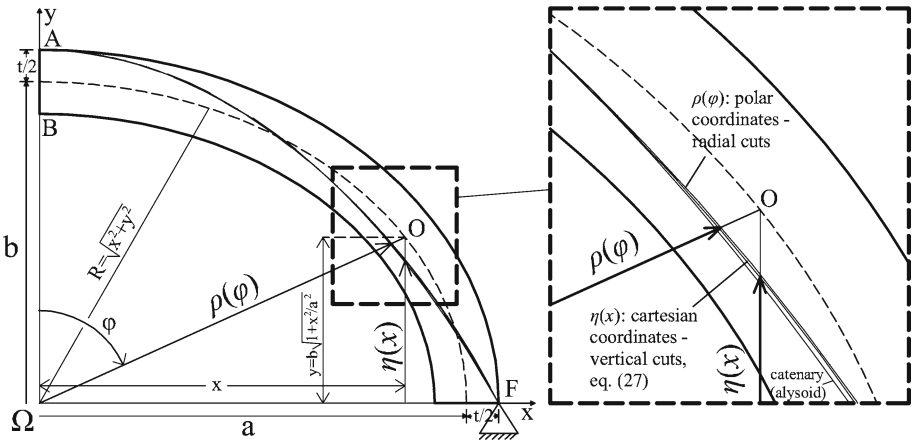


Fig. 2 Monolithic elliptical arch with the two different physically admissible minimum thrust-lines (*left*), $b/a = 0.75$ and $t/a = 0.15$. The two physically admissible thrust-lines ($\rho(\varphi)$ obtained with successive *radial cuts* and $\eta(x)$ obtained with successive *vertical cuts*) are distinguishable in the enlarged *parallelogram* (*right*). The catenary (alysoid) that passes by points A and F is not a physically admissible thrust-line

Figure 2 (left) plots these two different minimum thrust-lines within a monolithic elliptical arch with $b/a = 0.75$ and $t/a = 0.15$. They have been constructed with a custom-made computer code which repeats the force equilibrium shown in Fig. 1 as point O runs from the crown to the springing. These two physically admissible thrust-lines are not distinguishable in the scale of the arch; yet, if one zooms in the neighborhood isolated with the dashed parallelogram, the two thrust-lines are clearly different as shown in Fig. 2 (right). Both minimum thrust-lines are equally correct, and the fact that they lie within the physical boundaries of the arch ensures that the arch is stable.

In structural engineering, the derivation of multiple solutions from equilibrium analysis is not common. The "counterintuitive" result of having two different yet neighboring theoretically correct answers for the thrust-line depending on the coordinate system adopted derives from the request to express the load path in a two-dimensional structure with finite thickness, t, with the thrust-line—that is a concept inherent to a one-dimensional structure (the "hanging chain"). Interestingly, the inverted "hanging chain" (catenary) that passes from the extreme points A and F of the extrados of the arch shown in Fig. 2 offers a third line that is different from the two minimum thrust-lines—the one computed by taking radial cuts after adopting a polar coordinate system and the other computed with vertical cuts after adopting a cartesian coordinate system.

3 Major milestones regarding the minimum thickness of semicircular masonry arches

Couplet [14] gave a remarkably complete solution to the problem of computing the minimum thickness of a semicircular masonry arch which can support its own weight. His work brings forward the two key concepts needed in the limit equilibrium analysis of masonry arches: (a) the limiting thrust-line and (b) the imminent mechanism of collapse. Nevertheless, Couplet [14] does not compute the exact location of the intrados hinge of the mechanism and assumes that its location is at $\varphi_r = 45°$. In this way, Couplet forces the selection of the mechanism; therefore, his solution for the minimum thickness, $t/R = 0.101$, is unconservative given that a neighboring mechanism might have developed at a larger thickness. In the early nineteenth century, the Spanish engineer Joaquin Monasterio works on the problem of determining the minimum thickness of masonry arches. His in-depth study that was unknown until recently (Foce [15], Albuerne and Huerta [16]) concludes for the minimum thickness of the semicircular arch that $1/9 < t/R_{\text{in}} < 1/8$, where $R_{\text{in}} = R - t/2$. Monasterio's inequality for the minimum thickness of semicircular arches, $0.1053 < t/R < 0.1176$, improves Couplet's unconservative result, $t/R = 0.101$.

About one century after Monasterio's work, Milankovitch [17,18] presented a rigorous general formulation for the thrust-line of arches that do not sustain tension. For the specific case of symmetric semicircular arches, Milankovitch's formulation employs polar coordinates and yields the closed form expression given by Eq. (1) for the radius of curvature, $\rho(\varphi)$, of the thrust-line (see also Foce [22]).

The expression given by Eq. (1) does not assume a minimum thrust-line; therefore, ρ_o is the unknown distance from the center of the circular arch to the point of application of the unknown horizontal thrust, H,

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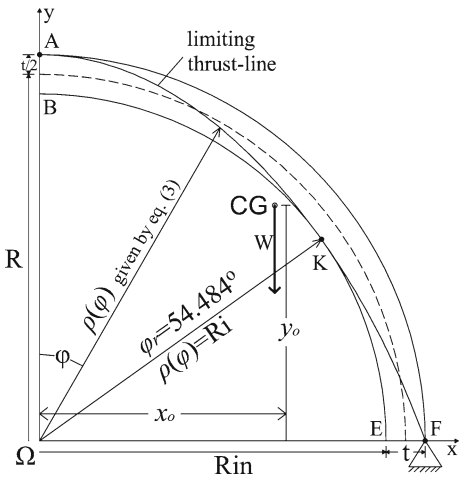


Fig. 3 Minimum thickness $t/R = 0.10748$ of a semicircular monolith as computed by Milankovitch [17,18] after adopting a polar coordinate system together with the minimum thrust-line given by Eq. (3) which touches the intrados at the rupture angle $\varphi_r = 54.484^\circ$

acting at the crown of the arch. Among all the admissible thrust-lines, the minimum thrust-line, as defined in the previous section of this paper, passes through the extrados points A ($\varphi = 0$) at the crown and F ($\varphi = \pi/2$) at the springing of the arch. These two boundary conditions are $\rho(0) = \rho_o = R + t/2$ and $\rho(\pi/2) = R + t/2$, which give the value of the minimum horizontal thrust-force,

$$H = tR\frac{3\pi(2+\frac{t}{R}) - (12+\frac{t^2}{R^2})}{6(2+\frac{t}{R})}. \tag{2}$$

By feeding back the result of Eq. (2) into Eq. (1) and after using that $\rho(0) = R + t/2$, Milankovitch [17,18] obtained a remarkable closed form expression for the radius of curvature $\rho(\varphi)$ of the minimum thrust-line:

$$\rho(\varphi) = R\frac{3\pi\left(2+\frac{t}{R}\right)^2 - \left(2+\frac{t}{R}\right)\left(12+\frac{t^2}{R^2}\right)\cos\varphi}{6\pi\left(2+\frac{t}{R}\right)\cos\varphi - 2\left(12+\frac{t^2}{R^2}\right)\cos\varphi + 12\left(2+\frac{t}{R}\right)\varphi\sin\varphi}. \tag{3}$$

Upon deriving the closed form expression for the radius of curvature, $\rho(\varphi)$, of the minimum thrust-line given by Eq. (3), Milankovitch [17,18] computes for the first time the correct solution for the theoretical minimum thickness, t, of a semicircular arch with radius R. When the circular arch assumes its minimum thickness, the minimum thrust-line given by Eq. (3) also touches the intrados of the arch at some point K which defines the unknown rupture angle $\varphi_r = \mathrm{B}\Omega\mathrm{K}$ as shown in Fig. 3. At the rupture angle, $\varphi_r = \mathrm{B}\Omega\mathrm{K}$, the radius of the thrust-line $\rho(\varphi)$ given by Eq. (3) reaches a minimum ($\mathrm{d}\rho(\varphi)/\mathrm{d}\varphi = 0$) and for this value of φ_r, $\rho(\varphi_r) = R - t/2$. With these two constraints, Milankovitch [17,18] finds $\varphi_r = \mathrm{B}\Omega\mathrm{K} = 54.484^\circ$ and that the minimum allowable thickness is $t/R = 0.10748$.

In principle, one could follow Milankovitch's formulation to compute the minimum thickness of elliptical arches with a geometric formulation. The main challenge, however, is that when a polar coordinate system is adopted, a closed form solution of the minimum thrust-line $\rho(\varphi)$ for the general elliptical arch is not known to the authors. Nevertheless, a solution to this problem when a polar coordinate system is adopted is possible by employing a variational formulation as was formulated recently for the limit equilibrium analysis of the semicircular arch (Makris and Alexakis [11,12], Alexakis [13]).

In the late 1960s, Heyman [7] revisits the problem of calculating the minimum thickness of circular masonry arches by adopting a polar coordinate system. Apparently, Heyman [7] was not aware of Milankovitch's [17,18] seminal work and attempted to solve the problem with a geometrical formulation by assuming incorrectly that the line of action of the resultant thrust-force is tangent to the intrados of the arch. Milankovitch's [17,18] correct results ($\varphi_r = 54^\circ 29'$ and $t/R = 0.1075$) remain nearly unknown in the literature (see Foce [22]), and Heyman's [7] unconservative solution ($\varphi_r = 58^\circ 49'$ and $t/R = 0.106$) becomes widely accepted (see for instance De Rubeis [24], Bićanić et al. [25] among others). Ochsendorf [19] confirms Milankovitch's [17,18]

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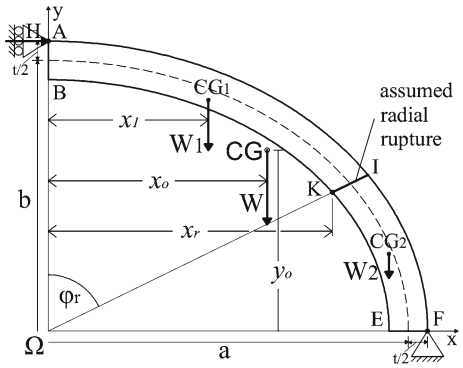


Fig. 4 Formation of a hinge mechanism in an *elliptical arch* by assuming that the rupture that initiates at the intrados hinge, K, happens along the radial direction which is consistent with a polar coordinate system

correct result by employing a "work balance" equation, and more recently, Heyman [20] acknowledged his subtle error.

4 Polar coordinates—variational formulation

Milankovitch [17,18] computed the correct value of the minimum thickness of the semicircular arched monolith, $t/R = 0.10748$, by following a geometric formulation and deriving the closed form expression of the minimum thrust-line given by Eq. (3) in association with the information that when the semicircular monolith assumes its minimum thickness, the minimum thrust-line given by (3) also touches the intrados of the arch.

When the thickness of the elliptical arch is sufficiently reduced and the minimum thrust-line touches the intrados, the arch reaches a limit equilibrium state by developing a five-hinge symmetric mechanism (Couplet [14]). Accordingly, points A, K, and F shown in Fig. 4 are imminent hinges of the arch at its limit equilibrium state.

Moment equilibrium of the hinged mechanism about the springing hinge, F, gives the value of the minimum horizontal thrust-force H at the extrados point of the crown

$$H = W\frac{3\pi\left(1+\frac{b}{a}\right)\left(2+\frac{t}{a}\right)-8\left(1+2\frac{b}{a}+\left(\frac{t}{2a}\right)^2\right)}{3\pi\left(1+\frac{b}{a}\right)\left(2\frac{b}{a}+\frac{t}{a}\right)} \tag{4}$$

in which $W = (a+b)t\pi/4$ is the weight of half the elliptical arch.

When assuming a rupture along the radial direction as shown in Fig. 4, moment equilibrium of the upper portion of the arch ABK about hinge K gives

$$H\left(b+\frac{t}{2}-y_{\text{in}}(x_r)\right) = W_1(x_r - x_1), \tag{5}$$

where H is the minimum horizontal thrust-force given by Eq. (4), and W_1 and x_1 are the weight and the abscissa of the center of gravity of the upper hinged portion of the elliptical arch given by

$$\begin{aligned} W_1 &= \int_0^{x_{\text{rex}}} y_{\text{ex}}(x) - y_p(x)\mathrm{d}x - \int_0^{x_r} y_{\text{in}}(x) - y_p(x)\mathrm{d}x \\ &= \frac{ab}{2}\left[c_{1a}c_{1b}\operatorname{arccsc}\left(\sqrt{1+\frac{a^2}{b^2}\frac{c_{1a}^2}{c_{1b}^2}\cot(\varphi)^2}\right) - c_{2a}c_{2b}\operatorname{arccsc}\left(\sqrt{1+\frac{a^2}{b^2}\frac{c_{2a}^2}{c_{2b}^2}\cot(\varphi)^2}\right)\right] \end{aligned} \tag{6}$$

and

$$x_1 = \frac{\int_0^{x_{\text{rex}}}\left(y_{\text{ex}}(x)-y_p(x)\right)x\mathrm{d}x - \int_0^{x_r}\left(y_{\text{in}}(x)-y_p(x)\right)x\mathrm{d}x}{W_1}. \tag{7}$$

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In Eqs. (5)–(7), the quantities $y_{\text{in}}(x)$ and $y_{\text{ex}}(x)$ are the ordinates of the intrados and extrados of the arch expressed in cartesian coordinates,

$$y_{\text{in}}(x) = \left(b - \frac{t}{2}\right)\sqrt{1 - \frac{x^2}{\left(a - \frac{t}{2}\right)^2}} = c_{2b}b\sqrt{1 - \frac{x^2}{(c_{2a}a)^2}}, \tag{8a}$$

$$y_{\text{ex}}(x) = \left(b + \frac{t}{2}\right)\sqrt{1 - \frac{x^2}{\left(a + \frac{t}{2}\right)^2}} = c_{1b}b\sqrt{1 - \frac{x^2}{(c_{1a}a)^2}}, \tag{8b}$$

and the integrals appearing in the numerator of Eq. (7) are

$$\int_0^{x_{\text{rex}}} \left(y_{\text{ex}}(x) - y_p(x)\right) x\,\mathrm{d}x = a^2 b c_{1a}^2 c_{1b} \frac{\sqrt{\frac{b^2}{a^2}c_{1b}^2 + c_{1a}^2\cot(\varphi)^2} - c_{1a}\cot(\varphi)}{3\sqrt{\left(\frac{b}{a} - 1\right)\left(\frac{b}{a} + \frac{t}{a} + 1\right) + c_{1a}^2\csc(\varphi)^2}}, \tag{9}$$

$$\int_0^{x_r} \left(y_{\text{in}}(x) - y_p(x)\right) x\,\mathrm{d}x = a^2 b c_{2a}^2 c_{2b} \frac{\sqrt{\frac{b^2}{a^2}c_{2b}^2 + c_{2a}^2\cot(\varphi)^2} - c_{2a}\cot(\varphi)}{3\sqrt{\left(\frac{b}{a} - 1\right)\left(\frac{b}{a} - \frac{t}{a} + 1\right) + c_{2a}^2\csc(\varphi)^2}}, \tag{10}$$

whereas the dimensionless quantities c_{1a}, c_{2a}, c_{1b}, and c_{2b} are given by

$$c_{1a} = 1 + \frac{1}{2}\frac{t}{a}, \quad c_{2a} = 1 - \frac{1}{2}\frac{t}{a}, \quad c_{1b} = 1 + \frac{1}{2}\frac{t}{b}, \quad c_{2b} = 1 - \frac{1}{2}\frac{t}{b}. \tag{11}$$

The ordinates $y_p(x)$ of the points along a polar cut at any angle φ are given by

$$y_p(x) = x\cot(\varphi), \tag{12}$$

and the solution of the equation

$$y_p(x) = y_{\text{in}}(x) \tag{13}$$

gives the abscissa, x_r, of the rupture point K at the intrados as a function of the angle φ

$$x_r = \frac{\sqrt{\left(b - \frac{t}{2}\right)^2\left(a - \frac{t}{2}\right)^2}}{\sqrt{\left(b - \frac{t}{2}\right)^2 + \left(a - \frac{t}{2}\right)^2\cot(\varphi)^2}} = a\frac{c_{2a}c_{2b}}{\sqrt{c_{2b}^2 + \frac{a^2}{b^2}c_{2a}^2\cot(\varphi)^2}}. \tag{14}$$

Similarly, the equation

$$y_p(x) = y_{\text{ex}}(x) \tag{15}$$

gives the abscissa, x_{rex}, of the point I at the extrados as a function of the angle φ:

$$x_{\text{rex}} = \frac{\sqrt{\left(b + \frac{t}{2}\right)^2\left(a + \frac{t}{2}\right)^2}}{\sqrt{\left(b + \frac{t}{2}\right)^2 + \left(a + \frac{t}{2}\right)^2\cot(\varphi)^2}} = a\frac{c_{1a}c_{1b}}{\sqrt{c_{1b}^2 + \frac{a^2}{b^2}c_{1a}^2\cot(\varphi)^2}}. \tag{16}$$

With the expressions offered by Eqs. (14) and (16), the expression of W_1 and x_1 given by Eqs. (6) and (7) are in terms of the polar variable φ.

Substituting the expression of the minimum horizontal force H at the crown given by Eq. (4) together with Eqs. (6) and (7), Eq. (5) becomes a transcendental equation that relates the unknown rupture location φ_r with the unknown minimum thickness t/a of any elliptical arch with a given height-to-span ratio b/a. Numerical solution of Eq. (5) yields the graph shown in Fig. 5, which offers the admissible values of t/a for a given value of the rupture location φ_r. Thus, the graph of Fig. 5 defines a functional relation

$$\frac{t}{a} = f\left(\varphi_r, \frac{b}{a}\right) \tag{17}$$

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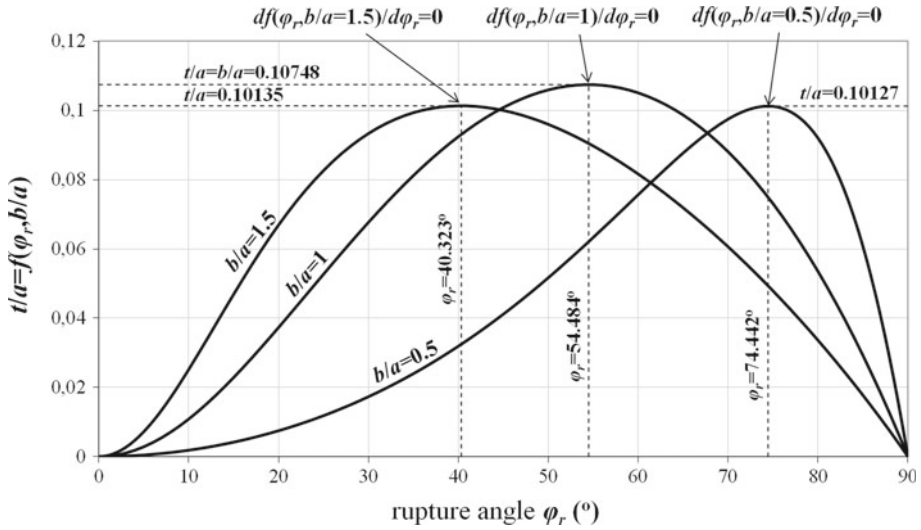


Fig. 5 Admissible values of $t/a = f(\varphi_r, b/a)$ of an elliptical arch with a given b/a for any potential rupture angle φ_r as they result from moment equilibrium of the hinged mechanism shown in Fig. 4. The principle of stationary potential energy ($\delta V(\varphi_r, b/a) = 0$) indicates that the hinged *elliptical arch* with a radial rupture is in equilibrium when $\mathrm{d}f(\varphi_r, b/a) = 0$. For the three values of $b/a = 0.5$, 1, and 1.5 shown, $\delta V(\varphi_r, b/a)$ vanishes when $\varphi_r = 74.442°$, 54.484°, and 40.323°, and the corresponding minimum thickness values are $t/a = 0.1013$, 0.1075, and 0.1014

between the admissible values t/a and the potential rupture location φ_r. With reference to Fig. 4, the coordinates of the center of gravity of half the elliptical arch can be calculated as

$$x_o = \frac{4a(a+2b)+t^2}{3\pi(a+b)} \tag{18}$$

and

$$y_o = \frac{4b(2a+b)+t^2}{3\pi(a+b)}. \tag{19}$$

Adopting as a reference level the horizontal axis x ($y = 0$), the potential energy of the elliptical arch is

$$V = W \cdot y_o = a^3 \frac{t}{a}\left[\frac{1}{3}\frac{b}{a}\left(2+\frac{b}{a}\right)+\frac{1}{12}\frac{t^2}{a^2}\right]. \tag{20}$$

With the functional relation given by Eq. (17) being established with the graph shown in Fig. 5, the potential energy of the elliptical arch given by Eq. (20) can be expressed as a function of φ_r:

$$V\left(\varphi_r, \frac{b}{a}\right) = a^3 f\left(\varphi_r, \frac{b}{a}\right)\left[\frac{1}{3}\frac{b}{a}\left(2+\frac{b}{a}\right)+\frac{1}{12}f\left(\varphi_r, \frac{b}{a}\right)^2\right]. \tag{21}$$

The structural system shown in Fig. 4 is a typical case where the only forces (weights) acting in the system are conservative and where the work of all forces is accounted by the potential energy $V(\varphi_r, b/a)$ given by Eq. (21). According to the principle of stationary potential energy, the geometrically admissible hinged mechanism shown in Fig. 4 is in an equilibrium state if and only if the total potential energy is stationary, i.e.,

$$\delta V\left(\varphi_r, \frac{b}{a}\right) = \frac{\mathrm{d}V\left(\varphi_r, \frac{b}{a}\right)}{\mathrm{d}\varphi_r}\delta\varphi_r = 0. \tag{22}$$

Substitution of Eq. (21) into Eq. (22) gives

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_r}\left[f\left(\varphi_r, \frac{b}{a}\right)\left[\frac{1}{3}\frac{b}{a}\left(2+\frac{b}{a}\right)+\frac{1}{12}f\left(\varphi_r, \frac{b}{a}\right)^2\right]\right] = 0, \tag{23}$$

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Table 1 Minimum thickness t/a and rupture angle $\varphi_r = \mathrm{B\Omega K}$ (x_r/a) of elliptical arches with semispan $= a$ and height $= b$

	Polar coordinates			Cartesian coordinates			Minimum thickness of an elliptical arch needed to accommodate a catenary (alysoid). The catenary is not a physically admissible thrust-line; therefore, the location x_N (φ_N) where the catenary touches the intrados is not an imminent rupture location		
	Variational formulation (see Sect. 4)			Values from the geometric formulation (see Sect. 5) are identical to the values from the variational formulation (see Sect. 6)					
b/a	Minimum thickness: t/a	Rupture angle: $\varphi_r(^\circ) = \mathrm{B\Omega K}$	Rupture location: x_r/a	Minimum thickness: t/a	Rupture angle: $\varphi_r(^\circ) = \mathrm{B\Omega K}$	Rupture location: x_r/a	t/a	$\varphi_N(^\circ)$	x_N/a
0.25	0.08273	84.123	0.86659	0.08422	84.101	0.86487	0.08029	84.213	0.87085
0.5	0.10127	74.442	0.81830	0.10340	74.533	0.81841	0.10242	75.067	0.82647
0.75	0.10709	64.015	0.78906	0.10925	64.299	0.79108	0.11045	65.443	0.80248
1	**0.10748**	**54.484**	**0.77021**	**0.10946**	**54.923**	**0.77359**	**0.11166**	**56.746**	**0.78956**
1.25	0.10513	46.567	0.75828	0.10689	47.090	0.76257	0.10928	49.570	0.78400
1.5	0.10135	40.323	0.75154	0.10291	40.875	0.75642	0.10502	43.951	0.78387
1.75	0.09684	35.516	0.74903	0.09821	36.067	0.75431	0.09986	39.670	0.78789
2	0.09200	31.857	0.75016	0.09324	32.395	0.75568	0.09439	36.446	0.79497

which simplifies to

$$\frac{\mathrm{d}f\left(\varphi_r, \frac{b}{a}\right)}{\mathrm{d}\varphi_r}\left[1 + \frac{1}{4}f\left(\varphi_r, \frac{b}{a}\right)^2\right] = 0. \tag{24}$$

The quantity in brackets in Eq. (24) is always positive; therefore, Eq. (24) is satisfied when $\mathrm{d}f(\varphi_r, b/a)/\mathrm{d}\varphi_r = 0$:

$$\frac{\mathrm{d}f\left(\varphi_r, \frac{b}{a}\right)}{\mathrm{d}\varphi_r} = \frac{\mathrm{d}(t/a)}{\mathrm{d}\varphi_r} = 0. \tag{25}$$

Numerical solution of the system of equations (5) and (25) gives the unknown location of the rupture angle φ_r and the minimum width t/a of an elliptical arch for a given height-to-semispan ratio b/a.

Table 1 (left columns) indicates the minimum thickness t/a and rupture angle $\varphi_r = \mathrm{B\Omega K}$ for various ratios of b/a as computed with the variational formulation developed in this section after adopting a polar coordinate system. For the case where $b/a = 1$ ($b = a = R$), the arch is a semicircle and the values $t/a = t/R = 0.10748$ and $\varphi_r = 54.484^\circ$ initially computed by Milankovitch [17,18] are recovered.

Figure 6 (top) plots with a solid line the minimum values of t/a, whereas Fig. 6 (center) plots the rupture angle φ_r (°) of elliptical arches, as computed after adopting a polar coordinate system. Figure 6 (top) indicates that when a polar coordinate system is adopted, the minimum thickness $t/a = 0.10776$ assumes its highest value for $b/a = 0.89503$ and the corresponding location of the imminent hinge is at $\varphi_r = 58.311^\circ$ ($x_r/a = 0.77714$). Next to the solid line, Fig. 6 (top) plots with a chain line the minimum thickness of elliptical arches that can just accommodate a catenary. The chain line is above the solid line for most of the range of b/a, except for shallow arches ($b/a < 0.4$), where the range is below. Accordingly, shallow arches ($b/a < 0.4$) may accommodate a catenary curve; yet, its thickness may be unconservative. On the other hand, shallow arches develop appreciable horizontal thrusts as shown in Fig. 6 (bottom) and their thickness may be governed by the compressive strength of the stone material. In summary, according to Fig. 6, the minimum thickness of an elliptical arch that is needed to accommodate the limiting hanging chain offers conservative values for most practical cases.

5 Cartesian coordinates—geometric formulation

5.1 A closed form expression of the minimum thrust-line of elliptical arches

Our analysis proceeds by adopting a cartesian coordinate system as shown in Fig. 1 (right) in association with the subtle piece of knowledge (Lamé and Clapeyron [26], Timoshenko [27], Milankovitch [17,18]) that the

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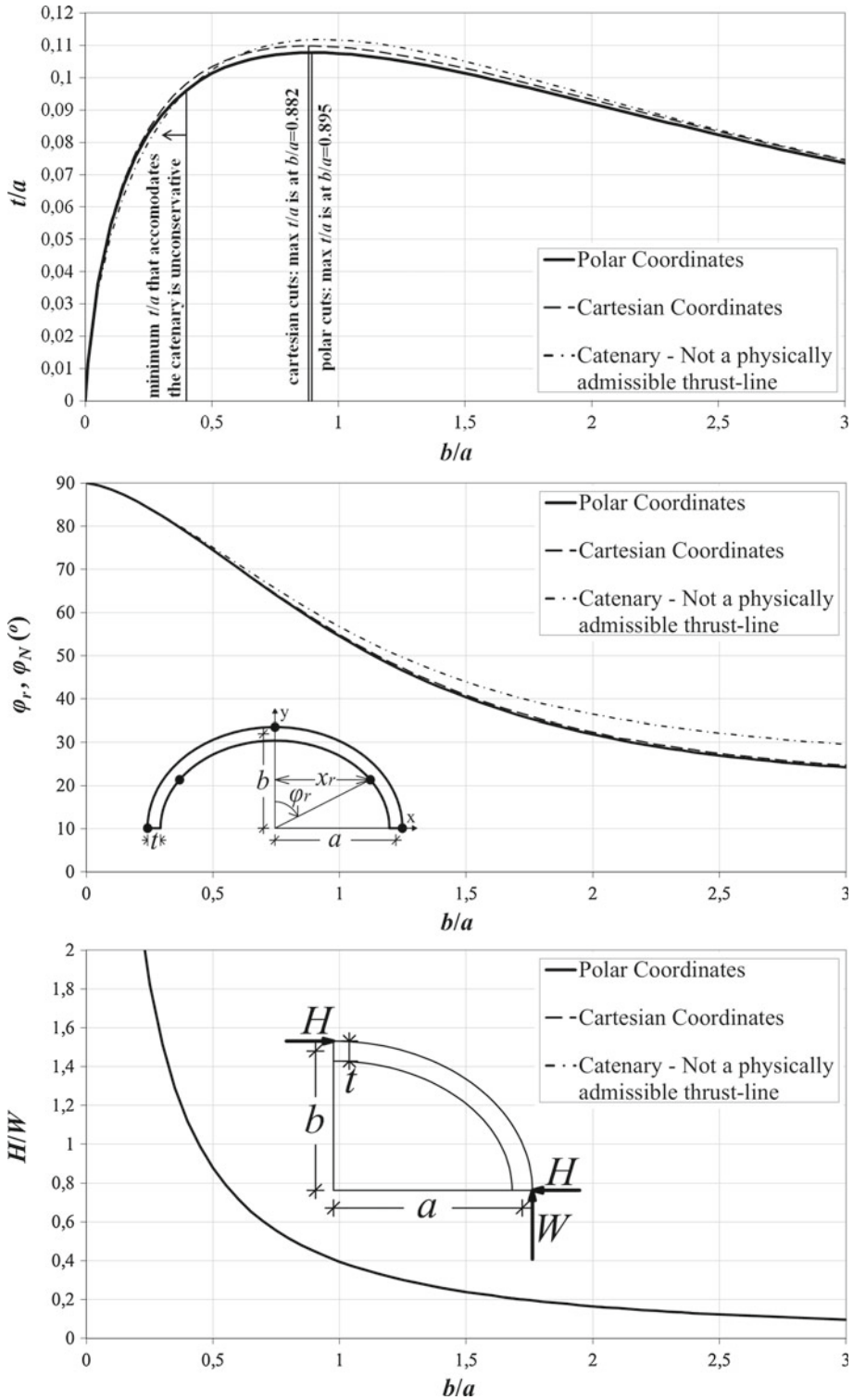


Fig. 6 Minimum thickness t/a (*top*) and location of the imminent intrados hinge (*center*) of elliptical masonry arches that do not sustain tension. The minimum thickness values, t/a, computed with a cartesian coordinate system (*vertical cuts*) are slightly higher than those computed with a polar coordinate system (*radial cuts*). The minimum horizontal thrust given by Eq. (4) is practically insensitive to the values of the minimum thickness t/a that results from the choice of the coordinate system (*bottom*)

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slope of the thrust-line when calculated with vertical cuts is $\mathrm{d}\eta(x)/\mathrm{d}x = \tan\psi_c = W(x)/H$, where $W(x)$ is the weight of the arch from the axis of symmetry ($x = 0$) up to the position x, and H is the horizontal thrust at the crown. Accordingly, starting from the axis of symmetry ($x = 0$) where the thrust-force, H, is horizontal, the vertical coordinate (ordinate) of the thrust-line at position x is given by

$$\eta(x) = \eta_o - \int_0^x \frac{W(x)}{H}\mathrm{d}x. \tag{26}$$

The expression given by Eq. (26) does not assume that the thrust-line is a minimum one; therefore, η_o is the unknown distance from the center of the elliptical arch to the point of application of the unknown horizontal thrust-force H acting at the crown of the arch. Among all the admissible thrust-lines that pass by point F ($x = a + t/2$) at the springing of the arch, the minimum thrust-line passes through the extrados point A ($x = 0$) at the crown (locations of the imminent hinge at the limit equilibrium state).

The first boundary condition gives $\eta_o = b + t/2$, whereas the minimum horizontal thrust, H, is given again by Eq. (4), and $W(x)$ is the weight of the arch from the axis of symmetry ($x = 0$) up to the position x. With these substitutions, Eq. (26) yields the ordinate of the minimum thrust-line

$$\eta(x) = b + \frac{t}{2} - \frac{y_{\mathrm{ex}}(x)\left(2c_{1a}^2\frac{a^2}{bt}+\frac{x^2}{bt}\right) - y_{\mathrm{in}}(x)\left(2c_{2a}^2\frac{a^2}{bt}+\frac{x^2}{bt}\right) - 2\frac{a^2}{t}\left(c_{1a}^2c_{1b}-c_{2a}^2c_{2b}\right) + 3x\frac{a}{t}\left[\frac{\pi}{2}\left(c_{1a}c_{1b}-c_{2a}c_{2b}\right) - c_{1a}c_{1b}\arccos\left(\frac{x}{c_{1a}a}\right) + c_{2a}c_{2b}\arccos\left(\frac{x}{c_{2a}a}\right)\right]}{\frac{3\pi\left(1+\frac{a}{b}\right)\left(1+\frac{t}{2a}\right) - \left(8+4\frac{a}{b}+\frac{t^2}{ab}\right)}{2\frac{b}{a}+\frac{t}{a}}}$$

$$\text{for } 0 \le x \le a - t/2, \tag{27a}$$

$$\eta(x) = \eta\left(a - \frac{t}{2}\right) - \frac{y_{\mathrm{ex}}(x)\left(2c_{1a}^2\frac{a^2}{bt}+\frac{x^2}{bt}\right) - 3a\left[\frac{a}{t}c_{2a}\left[\frac{\pi}{2}\left(c_{1a}c_{1b}-c_{2a}c_{2b}\right) - c_{1a}c_{1b}\arccos\left(\frac{c_{2a}}{c_{1a}}\right)\right] + \frac{\sqrt{2}}{3}\frac{a}{t}\sqrt{\frac{a}{t}}\left(2c_{1a}c_{1b} + \frac{c_{1b}c_{2a}^2}{c_{1a}}\right)\right] + 3x\frac{a}{t}\left[\frac{\pi}{2}\left(c_{1a}c_{1b}-c_{2a}c_{2b}\right) - c_{1a}c_{1b}\arccos\left(\frac{x}{c_{1a}a}\right)\right]}{\frac{3\pi\left(1+\frac{a}{b}\right)\left(1+\frac{t}{2a}\right) - \left(8+4\frac{a}{b}+\frac{t^2}{ab}\right)}{2\frac{b}{a}+\frac{t}{a}}}$$

$$\text{for } a - t/2 \le x \le a + t/2. \tag{27b}$$

In Eqs. (27a,b), the quantities $y_{\mathrm{in}}(x)$ and $y_{\mathrm{ex}}(x)$ are the ordinates of the extrados and intrados of the arch given by Eqs. (8a,b). Figure 2 plots the graph of Eq. (27), which is in full agreement with the value of the minimum thrust-line calculated with the in-house software by selecting a cartesian coordinate system and successive vertical cuts.

5.2 Minimum thickness of elliptical masonry arches using a geometric formulation

Equation (27) offers the ordinates of the minimum thrust-line of an elliptical arch that one computes after adopting a cartesian coordinate system. When the elliptical arch assumes its minimum thickness, the minimum thrust-line given by (27) also touches the intrados of the arch at some point K which defines the unknown rupture angle $\varphi_r = \mathrm{B}\Omega\mathrm{K}$ (see Fig. 7). Accordingly, with reference to Fig. 7, we are in search for the abscissa x_r for which

$$\eta(x_r) = y_{\mathrm{in}}(x_r), \tag{28}$$

where $\eta\,(x)$ is the minimum thrust-line given by Eq. (27a), and $y_{\mathrm{in}}(x)$ is given by Eq. (8a). In addition to the unknown rupture, x_r, both Eqs. (27a) and (8a) contain the unknown quantity t/a. The second equation needed to determine t/a is that the thrust-line is tangent to the intrados at hinge K ($x = x_r$). Accordingly,

$$\frac{d\eta(x)}{\mathrm{d}x}|_{x=x_r} = \frac{\mathrm{d}y_{\mathrm{in}}(x)}{dx}|_{x=x_r}. \tag{29}$$

From the two equations (28) and (29) and any given ratio of the height-to-semispan b/a of the arch, one solves for the two unknowns t/a and x_r/a. Table 1 offers next to the values computed with a polar coordinate system and a variational formulation, the minimum thickness values t/a and rupture abscissa x_r/a of elliptical arches computed with a geometric formulation after adopting a cartesian coordinate system. Note that the minimum thickness values t/a computed after adopting a cartesian coordinate system are slightly higher than those computed with a polar coordinate system, while the differences in the angle of rupture φ_r are marginal as shown in Fig. 6.

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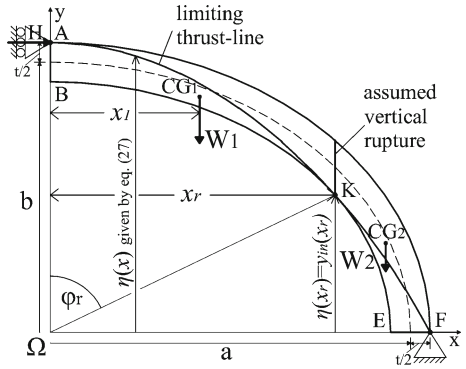


Fig. 7 Formation of a hinge mechanism in an *elliptical arch* as the thrust-line computed with a cartesian coordinate system touches the intrados of the *arch* at point K. The *rupture* that initiates at the intrados hinge K happens along the vertical direction which is consistent with the cartesian coordinate system

6 Cartesian coordinates—variational formulation

The minimum thickness values and rupture location of elliptical arches computed in the previous section with a geometric formulation after adopting a cartesian coordinate system are confirmed in this section with a variational formulation. We consider again that the elliptical arch has reached its limit equilibrium state by developing a five-hinge mechanism. Accordingly, points A, K, and F shown in Fig. 7 are imminent hinges of the arch at its limit equilibrium state. We now assume that the rupture happens along the vertical direction as shown in Fig. 7 (that is consistent with a cartesian coordinate system). For a vertical rupture, the weight W_1 of the upper hinged portion of the elliptical arch ABK is

$$\begin{aligned} W_1 &= \int_0^{x_r} y_{\text{ex}}(x)\mathrm{d}x - \int_0^{x_r} y_{\text{in}}(x)\mathrm{d}x \\ &= \frac{ab}{2}\left[c_{1a}c_{1b}\left(\frac{x_r}{ac_{1a}^2}\sqrt{c_{1a}^2-\frac{x_r^2}{a^2}}+\arcsin\left(\frac{x_r}{ac_{1a}}\right)\right)-c_{2a}c_{2b}\left(\frac{x_r}{ac_{2a}^2}\sqrt{c_{2a}^2-\frac{x_r^2}{a^2}}+\arcsin\left(\frac{x_r}{ac_{2a}}\right)\right)\right], \end{aligned} \tag{30}$$

while the abscissa x_1 of the center of gravity CG_1 of the upper hinged portion ABK can be expressed as

$$x_1 = \frac{\int_0^{x_r} y_{\text{ex}}(x)x\mathrm{d}x - \int_0^{x_r} y_{\text{in}}(x)x\mathrm{d}x}{W_1}, \tag{31}$$

where $y_{\text{in}}(x)$ and $y_{\text{ex}}(x)$ are given by Eqs. (8a) and (8b) and

$$\int_0^{x_r} y_{\text{ex}}(x)x\mathrm{d}x = \frac{1}{3}a^2bc_{1a}c_{1b}\left[c_{1a}+\left(\frac{x_r^2}{a^2}\frac{1}{c_{1a}}-1\right)\sqrt{c_{1a}^2-\frac{x_r^2}{a^2}}\right], \tag{32}$$

$$\int_0^{x_r} y_{\text{in}}(x)x\mathrm{d}x = \frac{1}{3}a^2bc_{2a}c_{2b}\left[c_{2a}+\left(\frac{x_r^2}{a^2}\frac{1}{c_{2a}}-1\right)\sqrt{c_{2a}^2-\frac{x_r^2}{a^2}}\right]. \tag{33}$$

In all Eqs. (30)–(33), x_r is the unknown abscissa of the rupture of the arch, while c_{1a}, c_{2a}, c_{1b}, and c_{2b} are given by Eq. (11). With reference to Fig. 7, moment equilibrium of the upper portion of the arch ABK about hinge K gives

$$H\left(b+\frac{t}{2}-y_{\text{in}}(x_r)\right) = W_1(x_r-x_1), \tag{34}$$

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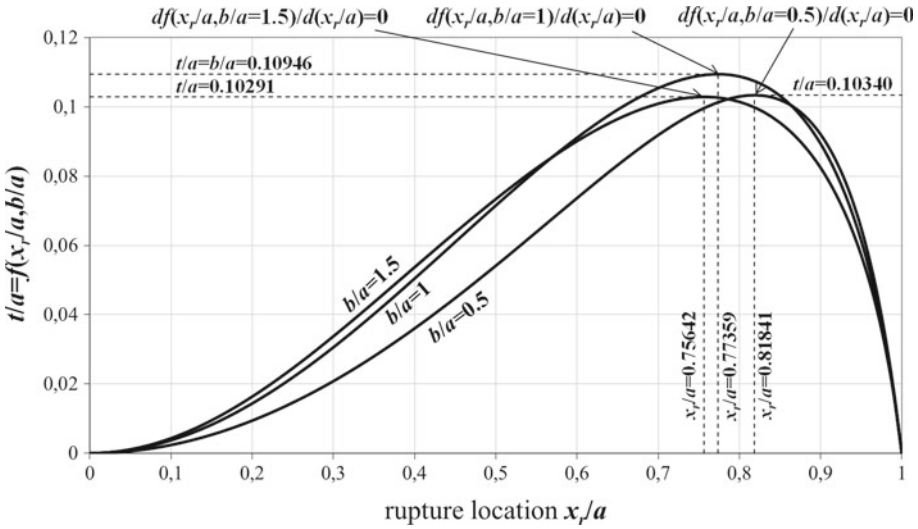


Fig. 8 Admissible values of $t/a = f(x_r/a, b/a)$ of an *elliptical arch* with a given b/a for any potential rupture location x_r/a as they result from moment equilibrium of the hinged mechanism shown in Fig. 7. The principle of stationary potential energy ($\delta V(x_r/a, b/a) = 0$) indicates that the hinged *elliptical arch* with a vertical rupture is in equilibrium when $\mathrm{d}f(x_r/a, b/a) = 0$. For the three values of $b/a = 0.5$, 1, and 1.5 shown, $\delta V(x_r/a, b/a)$ vanishes when $x_r = 0.8185$, 0.7736, and 0.7564, and the corresponding minimum thickness values are $t/a = 0.1034$, 0.1095, and 0.1029

while moment equilibrium of the entire half-arch about hinge F gives again the minimum horizontal thrust-force H at the crown given by Eq. (4).

Substitution of the expression of the minimum horizontal force H at the crown given by Eq. (4) together with the expressions for W_1 and x_1 given by Eqs. (30) and (31), Eq. (34) becomes a transcendental equation that relates the unknown rupture location x_r/a with the unknown minimum thickness t/a for any elliptical arch with a given height-to-semispan ratio b/a. Numerical solution of Eq. (34) yields the graph shown in Fig. 8, which offers the admissible values t/a for a given value of the rupture location x_r/a. Thus, the graph of Fig. 8 defines a functional relation

$$\frac{t}{a} = f\left(\frac{x_r}{a}, \frac{b}{a}\right) \tag{35}$$

between the admissible values of t/a and the potential rupture location x_r/a. With the functional relation $t/a = f(x_r/a, b/a)$ being established with the graph shown in Fig. 8, the potential energy of the elliptical arch given by Eq. (20) can be expressed as a function of x_r/a:

$$V\left(\frac{x_r}{a}, \frac{b}{a}\right) = a^3 f\left(\frac{x_r}{a}, \frac{b}{a}\right)\left[\frac{1}{3}\frac{b}{a}\left(2 + \frac{b}{a}\right) + \frac{1}{12} f\left(\frac{x_r}{a}, \frac{b}{a}\right)^2\right]. \tag{36}$$

According to the principle of stationary potential energy, the geometrically admissible hinged mechanism shown in Fig. 7 is in equilibrium state if and only if the total potential energy is stationary, i.e., $\delta V(x_r/a, b/a) = 0$.

The calculus that follows is identical to the calculus that has been presented in the case of polar coordinate system (Eqs. (23) and (24)); therefore, the symmetric elliptic arch with a vertical rupture at point K is in an equilibrium state ($\delta V(x_r/a, b/a) = 0$) if and only if

$$\frac{\mathrm{d}f\left(\frac{x_r}{a}, \frac{b}{a}\right)}{\mathrm{d}\frac{x_r}{a}} = \frac{\mathrm{d}\left(\frac{t}{a}\right)}{\mathrm{d}\frac{x_r}{a}} = 0. \tag{37}$$

Numerical solution of the system of Eqs. (34) and (37) gives the unknown abscissa of the internal hinge x_r/a and the minimum width t/a of an elliptical arch for a given height-to-semispan ratio b/a. The computed values are precisely the same values as those obtained with the geometrical formulation presented earlier and listed in Table 1.

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7 Conclusions

While the key concepts needed for the limit equilibrium analysis of arches were understood and published back in the early eighteenth century by Couplet [14], and the first complete and correct solution for the minimum thickness of semicircular arches has been published more than a century ago by Milankovitch [17,18], no results have been available for the minimum thickness of symmetric elliptical arches capable to support their own weight. Motivated from the recent growing interest in identifying the limit equilibrium states of historic structures, in this paper, we compute the minimum thickness together with the location of the imminent intrados hinge of symmetric elliptical masonry arches when subjected to their own weight.

The study first computes the minimum thrust-line of the elliptical arches that passes by the extrados points at the crown and the springing of the arch and shows that the profile of this minimum thrust-line depends on the coordinate system adopted. Accordingly, the paper presents two distinguishable physically admissible minimum thrust-lines, one after adopting a polar coordinate system and one after adopting a cartesian coordinate system. These two distinguishable physically admissible thrust-lines are neighboring thrust-lines to R. Hooke's catenary (alysoid) which is not a physically admissible thrust-line (Makris and Alexakis [11,12], Alexakis [13]).

Given the two above-mentioned limiting thrust-lines that depend on the stereotomy that is exercised on the arch, it is shown that the answer for the minimum thickness of symmetric masonry arches is not unique and that it depends on the coordinate system adopted. This result is confirmed in this work by developing a variational formulation after selecting the appropriate direction of the rupture that initiates at the intrados hinge of the arch. The variational formulation emerges from this work as a most powerful analysis tool since it is liberated from the concept of the thrust-line.

The paper concludes that R. Hooke's catenary (alysoid), although not a physically admissible thrust-line, where it can be accommodated within the profile of the arch offers a conservative value for the minimum arch thickness for most practical configurations.

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