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ORIGINAL

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The effect of stereotomy on the shape of the thrust-line and the minimum thickness of semicircular masonry arches

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Abstract More than a century ago, the Serbian engineer and astronomer Milutin Milankovitch presented a remarkable formulation for the thrust-line of arches that do not sustain tension, and by taking radial cuts and a polar coordinate system, he published for the first time the correct and complete solution for the theoretical minimum thickness, t, of a monolithic semicircular arch with radius R. This paper shows that Milankovitch's solution, $t/R = 0.1075$, is not unique and that it depends on the stereotomy exercised. The adoption of vertical cuts which are associated with a cartesian coordinate system yields a neighboring thrust-line and a different, slightly higher value for the minimum thickness ($t/R = 0.1095$) than the value computed by Milankovitch. This result has been obtained in this paper with a geometric and a variational formulation. The Milankovitch minimum thrust-line derived with radial stereotomy and our minimum thrust-line derived with vertical stereotomy are two distinguishable, physically admissible thrust-lines which do not coincide with R. Hooke's catenary that meets the extrados of the arch at the three extreme points. Furthermore, the paper shows that the catenary (the "hanging chain") is not a physically admissible minimum thrust-line of the semicircular arch, although it is a neighboring line to the aforementioned physically admissible thrust-lines. The minimum thickness of a semicircular arch that is needed to accommodate the catenary curve is $t/R = 0.1117$—a value that is even higher than the enhanced minimum thickness $t/R = 0.1095$ computed in this paper after adopting a cartesian coordinate system; therefore, it works toward the safety of the arch.

Keywords Stone arches · Limit equilibrium analysis · Line of resistance · Catenary curve · Variational formulation

1 Introduction

The art of building masonry arches with semicircular spans was advanced appreciably by the Romans through the extensive construction of bridges and aqueducts. Nevertheless, masonry arches and vaulted structures are also found much earlier in ancient Greece, mainly in the entrances of stadiums (Olympia, Nemea) and theaters (Filippi-Kavala)—see Fig. 1, while smaller arches found in Mesopotamia date much earlier [1]. In ancient Greek and Roman arches, the joints between the voussoirs are along the radial direction. The attractive element of radial joints in a circular arch is that it can lead to identical wedge-shaped voussoirs, while the joints are perpendicular to both the extrados and intrados of the arch. These are unique geometric characteristics that were decisive in ancient building practices given that builders of that time had no established rules for sizing the dimensions of the arch other than experience and esthetics.

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Fig. 1 Views of ancient Greek semicircular arches with radial joints. *Left*: entrance arch to the stadium of ancient Olympia. *Center*: the tunnel entrance of the stadium of ancient Nemea. *Right*: the entrance arch to the ancient theater of Filippi, Kavala

Hooke [2] was apparently the first to propose a rational rule for sizing masonry arches by describing the analogy in the load path between a "hanging chain," which forms a catenary in tension under its own weight, and a masonry arch which stands under compression. This analogy conceived by Hooke is expressed in the literature "As hangs the flexible line, so but inverted will stand the rigid arch" [3–5].

Due to its zero tensile strength, the masonry arch acts only in compression; therefore, according to R. Hooke's analogy, it is expected to satisfy equilibrium if its finite thickness can accommodate the shape of an inverted "hanging chain" which acts only in tension. The shape of the catenary (the "hanging chain") is a physically realizable load path which keeps the constituents of the one-dimensional chain (rings) in equilibrium. In the two-dimensional masonry arch, the physically realizable load path is the one defined by the points of application of the resultant forces that keep the voussoirs of the arch in equilibrium. This load path has been traditionally called the thrust-line (or the line of resistance, Moseley [6], or the druckkurve, Milankovitch [7,8]) and is defined as the geometrical locus of the application points of the resultant thrust-force that develops at any cross-section of the arch.

Given that the "hanging chain" is a one-dimensional structure and that the masonry arch with a finite thickness, t, is a two-dimensional structure, there is an infinite number of neighboring "hanging chains" that can be located within the masonry arch. As the thickness of the arch is reduced, the admissible thrust-lines are crowded together and when the minimum thickness, t, is reached, any physically admissible thrust-line shall pass by the imminent hinges that the arch will develop at the extrados points at the crown and at the springing of the arch. Accordingly, in this paper, we focus on the study of thrust-lines that pass by the extrados point F at the springing as shown in Fig. 2—as if the arch is supported by a hinge at point F. The minimum thrust-line for this configuration is the line that also passes by the extrados point A at the crown. In this paper, we first revisit the construction of the minimum thrust-line of masonry arches supported at the extrados point F of their springing and it is shown that its exact shape depends on the stereotomy exercised. The paper shows that the Milankovitch [7,8] minimum thrust-line derived by taking radial cuts and a polar coordinate system and our minimum thrust-line derived by taking vertical cuts and a cartesian coordinate system are two distinguishable, physically admissible lines of resistance, which do not coincide with R. Hooke's catenary that meets the extrados of the arch at the extreme points A and F.

When adopting R. Hooke's catenary curve as a candidate thrust-line (line of resistance) in a semicircular arch, there is also a silent, a priori commitment on the direction of the thrust-force acting at each location of the thrust-line, since by definition the thrust-force along a catenary has to be tangent to the catenary curve. This paper shows that this tangency commitment associated with the catenary line of resistance (which has been inherited from the statics of the one-dimensional "hanging chain") challenges the static equilibrium of masonry arches with finite thickness. Furthermore, the paper shows that the catenary curve is not a physically admissible minimum thrust-line, although it is a neighboring line to the aforementioned physically admissible thrust-lines.

The problem of determining the minimum thickness of masonry arches has challenged the engineering community since the early eighteenth century [9], was tackled with remarkable ingenuity by Monasterio in the early nineteenth century [10,11], was addressed rigorously in the early twentieth century in the nearly unknown work by Milankovitch [7,8], and remains worth discussing until today [12]. This paper shows that Milankovitch's solution for the minimum thickness $t/R = 0.1075$ is not unique and that it depends on the stereotomy exercised on the monolithic arch. The adoption of vertical cuts and the associated cartesian coordinate system yields a neighboring physically admissible thrust-line and a different, slightly higher value for the minimum thickness ($t/R = 0.1095$) than the one computed by Milankovitch. The paper explains that the "counterintuitive" result

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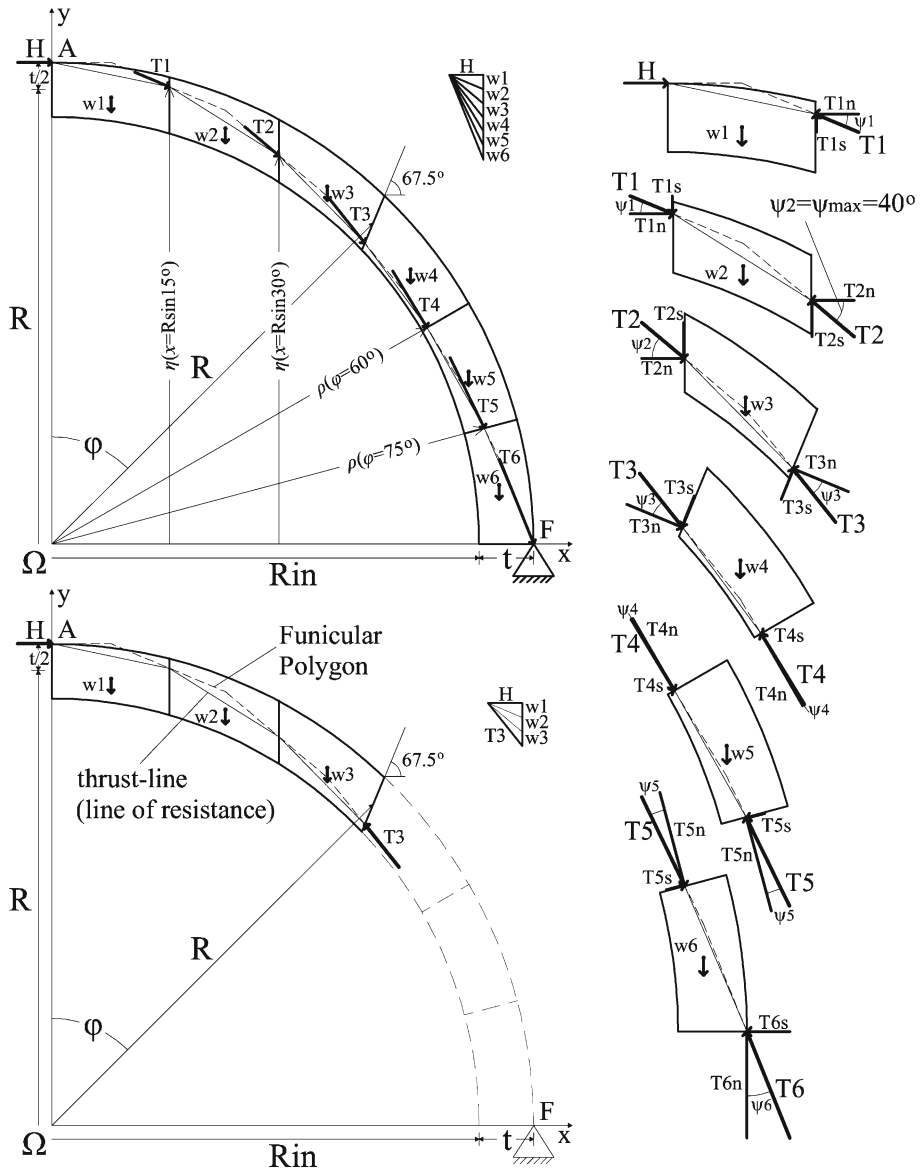


Fig. 2 Masonry arch with two vertical and two radial joints and $t/R = 0.12$. The *lines* of action of the resultant thrust-forces T_j generate the funicular polygon shown with a *dashed line*. *Top left*: The thrust-line (line of resistance shown with a *thin solid line*) is the geometrical locus of the points of application of the resultant thrust-forces T_j acting on the physical joints. *Bottom left*: Free-body diagram of the top portion of the arch together with the corresponding force polygon. *Right*: Resultant thrust-forces acting on the voussoirs together with the projections of the thrust-forces along the normal and shear directions at each joint. $\tan\psi_j = T_{js}/T_{jn}$

of having two different theoretically correct answers for the minimum thickness of masonry arches depending on the stereotomy exercised are inherently associated with the fact that a stable masonry arch has a finite thickness.

When a geometric formulation is adopted, the multiplicity in the solution for the minimum thickness t/R of the arch is a consequence from the "ambitious" request to express the load path in a two-dimensional arch with finite thickness with the thrust-line—that is a concept inherent to a one-dimensional structure (the "hanging chain").

When a variational formulation is adopted by examining the limiting equilibrium of the imminent hinge mechanism, the multiplicity in the solution for the minimum thickness, t/R, derives from the infinite many possible directions that the arch may rupture upon hinging has initiated (multiplicity in stereotomy). Each possible direction of the rupture defines distinct shapes/weights of the macroscopic articulated segments of the

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arch, and the minimum thickness t/R of the arch, as derived from the application of the principle of stationary potential energy, depends on the exact shape/weight of the articulated segments.

In view of the infinite many possible directions (different stereotomy) that an arch may rupture upon hinging has initiated; in this paper, in addition to the solution presented by Milankovitch [7,8] that is for a radial stereotomy, we present the solution for a vertical stereotomy that is better served with the adoption of a cartesian coordinate system.

In parallel with the macroscopic limit equilibrium analysis of the imminent five-hinge mechanism of the arch, one may conduct a finite element analysis of the arch by implementing the appropriate constitutive relations of the mechanical behavior of the masonry material [13–18]. Our paper revisits the limit equilibrium of masonry arches within the classical concepts of a macroscopic analysis of the limit state of the entire arch either with a geometric formulation (derivation of the limiting thrust-line) or with a variational formulation (application of the principle of stationary potential energy). The paper shows that the minimum thickness of a semicircular arch that is needed to accommodate the catenary curve (which is not a physically admissible thrust-line) is $t/R = 0.1117$—that is, a value even higher than the enhanced minimum thickness $t/R = 0.1095$ computed in this paper after adopting a cartesian coordinate system. Accordingly, Heyman's [19] solution remains unconservative regardless the stereotomy exercised on the arch—even if one assumes vertical joints (Heyman [12]).

Finally, the paper examines the additional thickness, ε, that is needed for the minimum thickness arch with finite strength to sustain the compressive stresses that develop under its own weight and concludes that the practical values of ε/R are two orders of magnitude less than the value of t/R. Therefore, the increased minimum thickness associated with the accommodation of R. Hooke's catenary works toward the safety of the arch and satisfies the limitations associated with the finite strength of masonry.

1.1 Background

When the masonry arch consists of individual voussoirs with finite dimensions, the thrust-line (line of resistance) is constructed by taking successive cuts of the arch along the physical joints of the voussoirs. For instance, consider the symmetric arch shown in Fig. 2 (top left), where $t/R = 0.12$. Starting from the vertical axis of symmetry ($\varphi = 0$), the first two voussoirs have vertical joints and the last two voussoirs have radial joints at $\varphi = 60°$ and $\varphi = 75°$, whereas the joint between the third and the fourth voussoir is neither vertical nor radial but at an angle of 67.5°. Figure 2 (right) shows the resultant thrust-forces acting on all six voussoirs of the masonry arch shown in Fig. 2 (top left) together with the projection of the thrust-forces along the normal and shear direction at the joints. The choice for selecting vertical joints between the first two voussoirs is unnatural; yet, it may be physically realizable as long as the friction coefficient along the masonry interface is sufficient ($\mu > \tan\psi_{\max} = T_{2s}/T_{2n} \approx 0.84$). This unnatural choice is intentional in an effort to set the stage for discussing later in the paper several subtle issues pertinent to the multiplicity of the minimum thrust-line and whether the thrust-forces are tangent to the thrust-line.

Figure 2 (bottom-left) shows the free-body diagram of a portion of the arch shown in Fig. 2 (top left) which consists of the three top voussoirs. The horizontal thrust, H, at the crown of the arch is applied at the extrados of the crown given that in the entire paper, we work with the minimum thrust-line as defined in the introduction of the paper. The joint at the right of the third voussoir is at an angle of 67.5° and the cut is along this angle. Next to the free-body diagram, the force polygon of the top portion of the arch expresses graphically the equilibrium of the system. The resultant thrust T_3 at joint 3 together with its point of application is uniquely determined from force and moment equilibrium (either graphically with the force polygon or analytically by solving equilibrium equations). This exercise is repeated at all six physical joints of the symmetric arch shown in Fig. 2 (top left) and the minimum thrust-line is uniquely defined. Together with the thrust-line (continuous thin line), Fig. 2 (top left) shows with bold line segments: (a) all the resultant thrust-forces acting on the physical joints as we move from the crown to the springing of the arch and (b) the weights of the individual voussoirs. The lines of action of the resultant thrusts generate the funicular polygon, and the lines of action of the weights of the voussoirs meet at the corners of the funicular polygon to satisfy moment equilibrium.

Figure 2 (top left) shows that the thrust-line (line of resistance = geometrical locus of the application points of the resultant forces that develops at the joints of the arch) and the funicular polygon are two neighboring lines and one may come to the conclusion that as we reduce the size of the voussoirs that these two lines will collapse to each other. Nevertheless, this is not always true and the collapse of these two lines to a single curve depends on the stereotomy exercised on the arch—that is, the directions of the physical joints when voussoirs with finite dimensions are used ([6–8,12,20], among others).

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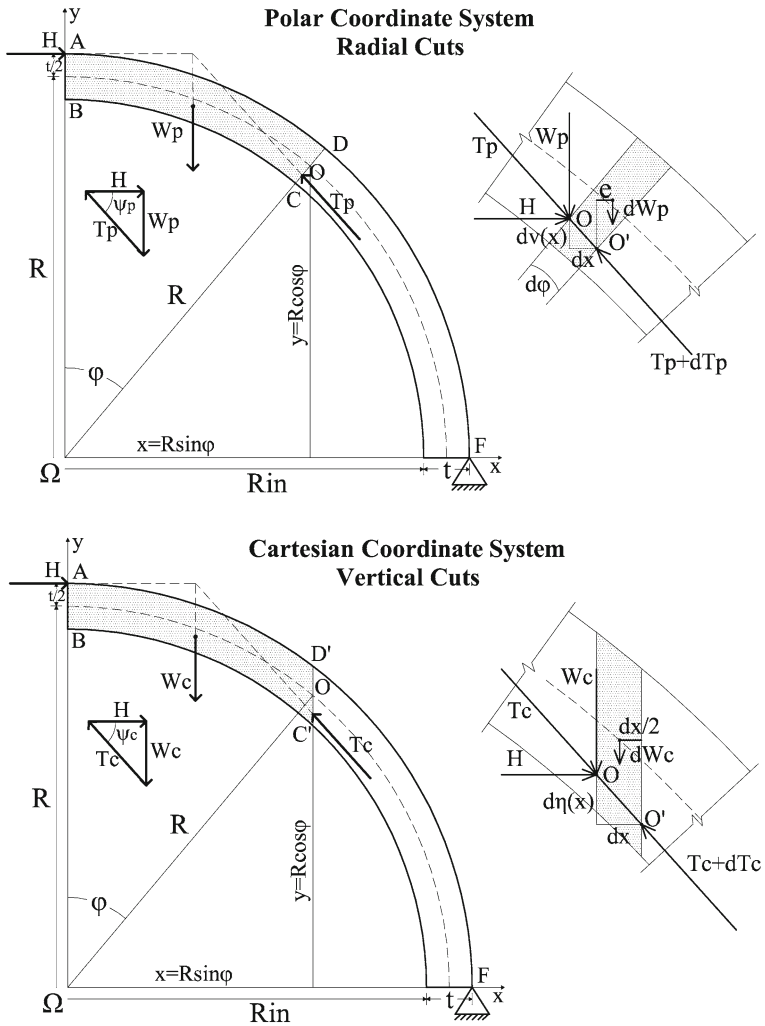


Fig. 3 Monolithic semicircular arch with zero tensile strength and $t/R = 0.12$. At any point O along the center line of the arch (*dashed line*) with polar coordinates R, φ or cartesian coordinates $x = R\sin\varphi$, $y = R\cos\varphi$, the thrust-force T_p along a radial cut after adopting a polar coordinate system (*top left*) is different than the thrust-force T_c along a vertical cut after adopting a cartesian coordinate system (*bottom left*). Accordingly, one obtains two distinguishable physically admissible minimum thrust-lines (see Fig. 4) depending on the coordinate system that is selected. Moment equilibrium of a slice of the monolith with radial cuts (*top right*) and vertical cuts (*bottom right*)

2 Thrust-lines of an arched monolith with zero tensile strength

In the previous section, it was shown that in the case where a symmetric arch consists of a finite number of voussoirs with physical joints, the minimum thrust-line is uniquely defined since equilibrium is taken on the existing physical joints. Consider now the monolithic semicircular arch shown in Fig. 3 which has zero tensile strength and therefore acts only in compression. Any point O that is located on the mid-thickness line of the arch (dashed line) has coordinates $x = R\sin\varphi$, $(0 \le x \le R)$ and $y = R\cos\varphi$ $(0 \le y \le R)$ since the angular coordinate φ starts measuring from the vertical axis of symmetry, y. We are interested in computing the resultant thrust-force acting within the arch at location O with coordinates $x = R\sin\varphi$ and $y = R\cos\varphi$ $(0 \le \varphi \le \pi/2)$. Standard way to proceed is to make a cut passing through point O and work the force and moment equilibrium as was shown in the previous section. One way to cut is along the radial direction after adopting a polar coordinate system as shown in Fig. 3 (top). With this choice, we isolate the portion of the arch ABCOD. Another way to cut is along the vertical direction after adopting a cartesian coordinate system as shown in Fig. 3 (bottom). With this choice, we isolate the portion of the arch AB-C'OD', which has different size and weight than the portion ABCOD; therefore, the thrust T_p along the

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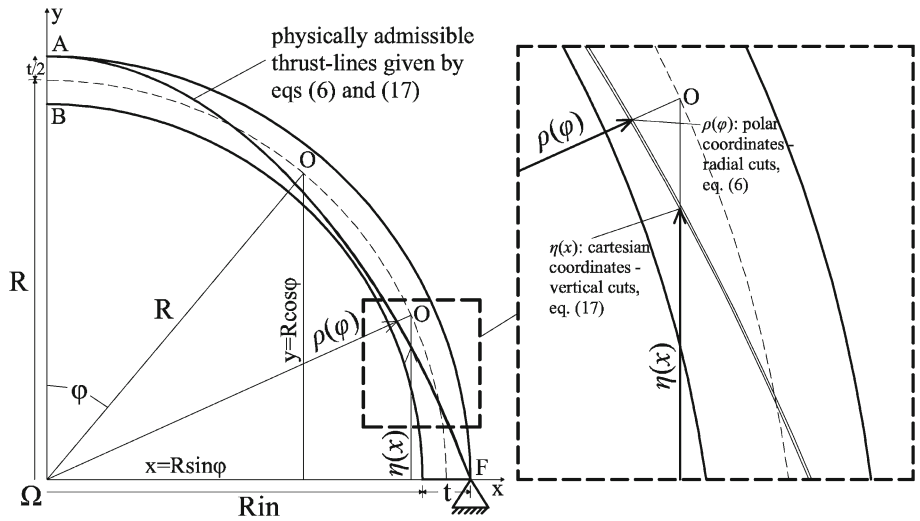


Fig. 4 Monolithic semicircular arch with the two different physically admissible minimum thrust-lines (*left*) and $t/R = 0.12$. The two physically admissible thrust-lines ($\rho(\varphi)$ obtained with successive radial cuts and $\eta(x)$ obtained with successive vertical cuts) are distinguishable in the enlarged parallelogram (*right*)

radial cut COD is different both in magnitude and direction than the thrust T_c along the vertical cut C'OD.' As this exercise is repeated for different points O along the mid-thickness line of the arch, one obtains two neighboring yet different thrust-lines—one thrust-line that results from radial cuts after adopting a polar coordinate system and a different thrust-line that results from vertical cuts after adopting a cartesian coordinate system.

Figure 4 (left) plots these two different minimum thrust-lines within a monolithic semicircular arch with $t/R = 0.12$. They have been constructed with a custom-made computer code which repeats the force equilibrium shown in Fig. 3 as point O runs from the crown to the springing. These two physically admissible thrust-lines are not distinguishable in the scale of the arch; yet, if one zooms in the neighborhood isolated with the dashed parallelogram, the two thrust-lines are clearly different as shown in Fig. 4 (right). Both minimum thrust-lines are equally correct and the fact that they lie within the physical boundaries of the arch ensures that the arch is stable.

In structural engineering, the derivation of multiple solutions from equilibrium analysis is not common. The "counterintuitive" result of having two different yet neighboring theoretically correct answers for the thrust-line depending on the stereotomy exercised and the associated coordinate system adopted derives from the request to express the load path in a two-dimensional structure with finite thickness, t, with the thrust-line—that is, a concept inherent to a one-dimensional structure (the "hanging chain").

Interestingly, the inverted "hanging chain" (catenary) that passes from the extreme points A and F of the extrados of the arch shown in Fig. 4 offers a third line that is different from the two minimum thrust-lines—the one computed by taking radial cuts after adopting a polar coordinate system (see Fig. 3 top) and the other computed with vertical cuts after adopting a cartesian coordinate system (see Fig. 3 bottom). This paper shows that the catenary curve is not a physically acceptable thrust-line of the semicircular arch.

In the event that the thickness, t, of the arch could be reduced continuously ($t \to 0$), the two aforementioned thrust-lines would eventually collapse to a single line since the size and weight of any portion of the arch ABO would tend to be independent of the direction of the cut. In masonry arches that do not sustain tension, this converging process is halted by the minimum finite thickness that is needed in a masonry arch to remain in equilibrium at its limit state under its own weight. Consequently, given that this converging process reaches an impasse ($t =$ finite), the two aforementioned thrust-lines and the catenary curve remain three distinguishable yet neighboring lines.

2.1 Radial stereotomy and polar coordinates: Milankovitch's minimum thrust-line equation

In his work of unparallel quality and rigor, Milankovitch [7,8] presents a general formulation for the thrust-line of curved masonry structures. For the specific case of symmetric circular arches (as the one shown in

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Fig. 3), Milankovitch's formulation employs radial cuts and after adopting a polar coordinate system, it yields a closed-form expression for the radius of curvature, ρ, of the thrust-line (see also [20])

$$\rho(\phi) = \frac{\rho_o H + \frac{1}{6} t\,(t^2 + 12R^2)\sin^2\frac{\phi}{2}}{H\cos\phi + tR\phi\sin\phi}. \tag{1}$$

It is worth noting that Milankovitch [7,8] obtains Eq. (1) with two different approaches: (a) via differential calculus and (b) via macroscopic equilibrium of a finite portion of the arch. Clearly, Eq. (1) is not the equation of a catenary; therefore, the questioning of whether the catenary is a physically admissible thrust-line goes back to the beginning of the twentieth century.

The expression given by Eq. (1) does not assume a minimum thrust-line; therefore, ρ_o is the unknown distance from the center of the circular arch to the point of application of the unknown horizontal thrust, H, acting at the crown of the arch. Among all the admissible thrust-lines, the minimum thrust-line, as defined in the introduction of the paper, passes through the extrados points A ($\varphi = 0$) at the crown and F ($\varphi = \pi/2$) at the springing of the arch. These two boundary conditions are

$$\rho(0) = \rho_o = R + \frac{t}{2} \tag{2}$$

and

$$\rho(\pi/2) = R + \frac{t}{2} \tag{3}$$

which give the value of the minimum horizontal thrust-force,

$$H = tR\frac{3\pi\left(2+\frac{t}{R}\right) - \left(12 + \frac{t^2}{R^2}\right)}{6\left(2+\frac{t}{R}\right)}. \tag{4}$$

Using that the weight of half the arch is $W = (\pi/2)tR$, the normalized minimum horizontal thrust is given by

$$\frac{H}{W} = 1 - \frac{1}{\pi}\frac{4 + \frac{1}{3}\frac{t^2}{R^2}}{2+\frac{t}{R}} \tag{5}$$

Figure 5 plots the graph of H/W as a function of t/R.

By feeding back the results of Eqs. (2) and (4) into Eq. (1), Milankovitch [7,8] obtained a remarkable closed-form expression for the radius of curvature ρ (φ) of the minimum thrust-line

$$\rho(\phi) = R\frac{3\pi\left(2+\frac{t}{R}\right)^2 - \left(2+\frac{t}{R}\right)\left(12+\frac{t^2}{R^2}\right)\cos\phi}{6\pi\left(2+\frac{t}{R}\right)\cos\phi - 2\left(12+\frac{t^2}{R^2}\right)\cos\phi + 12\left(2+\frac{t}{R}\right)\phi\sin\phi} \tag{6}$$

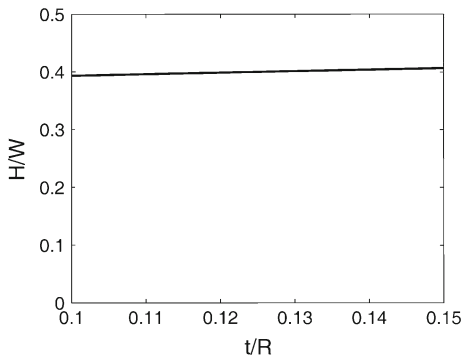


Fig. 5 Minimum horizontal thrust-force H at the crown of the arch normalized to the weight of half the arch, $W = (\pi/2)tR$, as a function of t/R

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Equation (6) offers the expression of the minimum thrust-line that passes by the extrados points A ($\varphi = 0$, crown) and F ($\varphi = \pi/2$, springing). Figure 4 plots the graph of Eq. (6) which is in full agreement with the value of the minimum thrust-line calculated with the in-house software by selecting a polar coordinate system and successive radial cuts.

Returning to Fig. 2 where the two bottom voussoirs have radial joints, the radial coordinates of the application points of the thrust-forces T_4 and T_5 are given precisely by Eq. (6). However, Eq. (6) does not offer the coordinates of the application points of thrust-forces T_1, T_2, and T_3 since the joints are not radial.

2.2 Direction of the line of action of the resultant thrust-force and the tangent to the thrust-line at the point of application of the thrust-force

At this point, it is worth mentioning the important conclusion derived by Milankovitch [7,8] that when a polar coordinate system is adopted (see Fig. 3 top), the slope of the thrust-force $\tan\psi_p = W_p/H$ at any given point O is in general different than the slope of the thrust-line $\mathrm{d}v(x)/\mathrm{d}x$ at that point O, where $v(x) = \rho(\varphi)\cos(\varphi)$ ($\varphi = \arcsin(x/R)$) is the vertical coordinate of Milankovitch's thrust-line given by Eq. (6) (see also [20]). This can be easily shown by taking two neighboring radial cuts at angles φ and $\varphi + d\varphi$ as shown in Fig. 3 (top right). By taking moment equilibrium at the application point O′ of the thrust-force $T_p + dT_p$, one obtains

$$-H\mathrm{d}v(x) + W_p\mathrm{d}x - \mathrm{d}W_p e = 0 \tag{7}$$

where e is the finite horizontal distance from point O′ to the center of gravity of the infinitesimal voussoir of the arch with weight dW_p. After dividing Eq. (7) with $H\mathrm{d}x$, one obtains

$$\tan\psi_p = \frac{W_p}{H} = \frac{\mathrm{d}v(x)}{\mathrm{d}x} + \frac{e}{H}\frac{\mathrm{d}W_p}{\mathrm{d}x} \tag{8}$$

The quantity $(e/H)\cdot(\mathrm{d}W_p/\mathrm{d}x)$ is a finite quantity; therefore, in the case where a polar coordinate system is adopted,

$$\tan\psi_p = \frac{W_p}{H} \neq \frac{\mathrm{d}v(x)}{\mathrm{d}x}. \tag{9}$$

The above non-equality shows that in the case of radial cuts, the line of action of the resultant thrust-force (funicular polygon) intersects the tangent to the thrust-line (line of resistance) = geometrical locus of the application points of the resultant thrust-force.

Milankovitch [7,8] further concludes that in the case of vertical cuts, one obtains the *Stützlinie* (see also [20]) and the non-equality expressed by (9) becomes an equality

$$\tan\psi_c = \frac{W_c}{H} = \frac{\mathrm{d}\eta(x)}{\mathrm{d}x} \tag{10}$$

where $\eta(x)$ is the ordinate of the thrust-line that one obtains after adopting a cartesian coordinate system.

This can be easily shown by taking two neighboring vertical cuts at abscissae x and $x + \mathrm{d}x$ as shown in Fig. 3 (bottom-right). By taking moment equilibrium at the application point O' of the thrust-force $T_p + \mathrm{d}T_p$, one obtains

$$-H\mathrm{d}\eta(x) + W_c\mathrm{d}x - \mathrm{d}W_c\frac{\mathrm{d}x}{2} = 0 \tag{11}$$

where $\mathrm{d}x/2$ is the lever arm of the infinitesimal weight $\mathrm{d}W_c$.

After dividing Eq. (11) with $H\mathrm{d}x$, one obtains

$$\tan\psi_c = \frac{W_c}{H} = \frac{\mathrm{d}\eta(x)}{\mathrm{d}x} + \frac{1}{2}\frac{\mathrm{d}W_c}{H} \tag{12}$$

The quantity $\mathrm{d}W_c/2H$ is a first-order differential that vanishes when compared with the other two finite quantities of Eqs. (12) and (10) is recovered.

The above analysis shows that when a polar coordinate system is adopted, the line of action of the resultant thrust-force is in general not tangent to the thrust-line given by Eq. (6), whereas when a cartesian coordinate system is adopted, the resultant thrust-force at any given point is tangent to the thrust-line given by Eq. (17).

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2.3 Vertical stereotomy and cartesian coordinates: an alternative minimum thrust-line equation

The idea of analyzing the stability of masonry arches by taking vertical cuts rather than radial cuts goes back to the seminal work of Lamé and Clapeyron [21], who showed that for symmetrical arches of any shape, the calculation of the position of the intrados hinge can be greatly simplified if instead of radial cross-sections, vertical cross-sections are contemplated (see also [22]).

With reference to Fig. 3 (bottom), we now take vertical cuts and we adopt the associated cartesian coordinate system. Given the result of Eq. (12), the slope of the thrust-line is $\mathrm{d}\eta(x)/\mathrm{d}x = \tan\psi_c = W(x)/H$, where $W(x)$ is the weight of the arch from the axis of symmetry ($x = 0$) up to the position x and H is the horizontal thrust at the crown. Accordingly, starting from the axis of symmetry ($x = 0$) where the thrust-force is horizontal, the vertical coordinate of the thrust-line at position x is given by

$$\eta(x) = \eta_o - \int_0^x \frac{W(x)}{H}\mathrm{d}x \tag{13}$$

The expression given by Eq. (13) does not assume a minimum thrust-line; therefore, η_o is the unknown distance from the center of the circular arch to the point of application of the unknown horizontal thrust-force H acting at the crown of the arch. Among all the admissible thrust-lines that pass by point F ($x = R + t/2$) at the springing of the arch, the minimum thrust-line passes through the extrados points A ($x = 0$) at the crown (locations of imminent hinges at the limit equilibrium state). The first boundary condition gives $\eta_o = R + t/2$, and the minimum horizontal thrust, H, is given again by Eq. (4). Substituting these values into Eq. (13), the ordinate of the minimum thrust-line is given by

$$\eta(x) = R + \frac{t}{2} - \frac{\int_0^x W(x)\mathrm{d}x}{W \cdot \left(1 - \frac{1}{\pi}\frac{4+\frac{1}{3}\frac{t^2}{R^2}}{2+\frac{t}{R}}\right)} \tag{14}$$

where the weight of the circular arch up to position x is given by

$$W(x) = W\frac{1}{\pi}\left[\pi + \frac{x}{t}\left(\sqrt{c_1^2 - \frac{x^2}{R^2}} - \sqrt{c_2^2 - \frac{x^2}{R^2}}\right) - \frac{R}{t}\left(c_1^2 \arccos\left(\frac{x}{Rc_1}\right) + c_2^2 \arccos\left(\frac{x}{Rc_2}\right)\right)\right]$$
$$\text{for } 0 \le x \le R - t/2 \tag{15a}$$

$$W(x) = W\frac{1}{\pi}\left(\pi + \frac{x}{t}\sqrt{c_1^2 - \frac{x^2}{R^2}} - \frac{R}{t}c_1^2 \arccos\left(\frac{x}{Rc_1}\right)\right)$$
$$\text{for } R - t/2 \le x \le R + t/2. \tag{15b}$$

In Eq. (15), the dimensionless quantities c_1, c_2 are

$$c_1 = 1 + \frac{1}{2}\frac{t}{R},\ c_2 = 1 - \frac{1}{2}\frac{t}{R}, \tag{16}$$

while $W = (\pi/2)tR$ is the weight of half the circular arch. Substitution of the expression for $W(x)$ given by Eqs. (15a) and (15b) into Eq. (14) gives a closed-form expression for the ordinate of the minimum thrust-line of circular arches expressed in cartesian coordinates

$$\eta(x) = R + \frac{t}{2} - \frac{y_{ex}(x)\left(2c_1^2\frac{R}{t} + \frac{x^2}{Rt}\right) - y_{in}(x)\left(2c_2^2\frac{R}{t} + \frac{x^2}{Rt}\right) - 2\frac{R^2}{t}\left(c_1^3 - c_2^3\right) + 3x\frac{R}{t}\left(\pi\frac{t}{R} - c_1^2\arccos\left(\frac{x}{Rc_1}\right) + c_2^2\arccos\left(\frac{x}{Rc_2}\right)\right)}{3\pi - \left(12 + \frac{t^2}{R^2}\right)/\left(2 + \frac{t}{R}\right)}$$
$$\text{for } 0 \le x \le R - t/2 \tag{17a}$$

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$$\eta(x)=\eta\left(R-\frac{t}{2}\right)$$
$$-\frac{y_{\mathrm{ex}}(x)\left(2c_1^2\frac{R}{t}+\frac{x^2}{Rt}\right)-3R\left[c_2\left(\pi-c_1^2\frac{R}{t}\arccos\left(\frac{c_2}{c_1}\right)\right)+\sqrt{2}\frac{R}{t}\sqrt{\frac{t}{R}}\left(c_1^2-\frac{2}{3}\frac{t}{R}\right)\right]+3x\frac{R}{t}\left(\pi\frac{t}{R}-c_1^2\arccos\left(\frac{x}{Rc_1}\right)\right)}{3\pi-\left(12+\frac{t^2}{R^2}\right)/\left(2+\frac{t}{R}\right)}$$
for $R-t/2\le x\le R+t/2$. (17b)

In Eqs. (17a) and (17b), the quantities $y_{\mathrm{ex}}(x)$ and $y_{\mathrm{in}}(x)$ are the ordinates of the extrados and intrados of the arch expressed in cartesian coordinates

$$y_{\mathrm{ex}}(x)=R\sqrt{\left(1+\frac{1}{2}\frac{t}{R}\right)^2-\frac{x^2}{R^2}}=R\sqrt{c_1^2-\frac{x^2}{R^2}} \tag{18a}$$

and

$$y_{\mathrm{in}}(x)=R\sqrt{\left(1-\frac{1}{2}\frac{t}{R}\right)^2-\frac{x^2}{R^2}}=R\sqrt{c_2^2-\frac{x^2}{R^2}}. \tag{18b}$$

Figure 4 plots the graph of Eq. (17) which is in full agreement with the value of the minimum thrust-line calculated with the in-house software by selecting successive vertical cuts.

Returning to Fig. 2, where the two top voussoirs have vertical joints, the vertical coordinates of the application points of the thrust-forces T_1 and T_2 are given precisely by Eq. (17a). Accordingly, the application points 1 and 2 of the thrust-forces that develop along the masonry arch shown in Fig. 2 (top) belong to the thrust-line given by Eq. (17a) while the application points 4 and 5 belong to the thrust-line given by Eq. (6).

In the foregoing analysis, we derived a physically admissible Eq. (17a and 17b) for the minimum thrust-line of an arched monolith which has zero tensile strength after adopting a cartesian coordinate system. Equation (17) offers a thrust-line that is slightly different than the physically admissible minimum thrust-line derived by Milankovitch [7,8] given by Eq. (6) who adopted a polar coordinate system. Nevertheless, neither Eq. (17) nor Eq. (6) are expressions of the catenary—that is, the shape of the "hanging chain" that passes through the extrados points A($x=0$) at the crown and F($x=R+t/2$) at the springing of the arch. This finding raises the question whether the catenary curve is a physically admissible curve for the thrust-line of the circular arch.

2.4 Can the catenary (the "hanging chain") be a physically admissible minimum thrust-line of the semicircular arch?

The fundamental property of the catenary (alysoid) is that at any location of the curve, the internal tensile force is along the tangent of the curve at this location. Its general mathematical form is that of a hyperbolic cosine. Herein, we are interested in creating a minimum thrust-line of the semicircular arch, so that a chain with uniform weight has to be pinned at the springing point F ($x=R+t/2=R_{\mathrm{ex}}$) of the arch and to be long enough so that its maximum (summit) to reach the extrados point A at the crown of the arch ($h(0)=R+t/2=R_{\mathrm{ex}}$)). With the last boundary condition, the equation of the catenary curve assumes the expression

$$h(x)=R_{\mathrm{ex}}\left[1+a\left(1-\cosh\left(\frac{x}{aR_{\mathrm{ex}}}\right)\right)\right] \tag{19}$$

where the value of parameter $\alpha=0.61876$ is determined from the first boundary condition $h(x=R_{\mathrm{ex}})=0$. Figure 6 (left) shows a semicircular monolith with zero tensile strength and $t/R=0.12$ together with the graph of the catenary expressed by Eq. (19). Equation (19) was not constructed (as in the previous section) by taking equilibrium of any portion of the arch. The only physical requirement is that at any location of the catenary curve [given by Eq. (19)], the internal thrust-force to be along the tangent of the catenary curve at this location. Below, we show that this tangency constraint together with the need that any portion of the circular arch has to be in equilibrium cannot be satisfied simultaneously; therefore, the catenary (the "hanging chain") is not a physically acceptable thrust-line that passes through points A and F.

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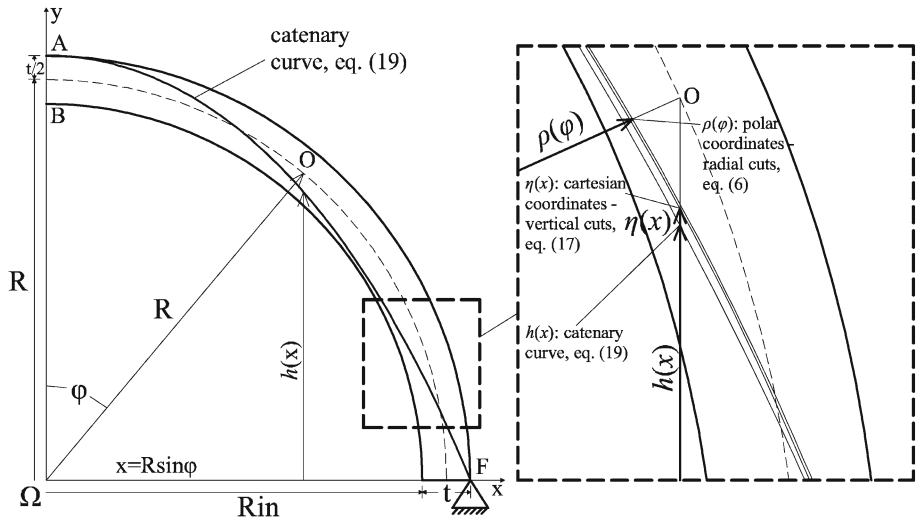


Fig. 6 *Left*: Semicircular arched monolith with $t/R = 0.12$ which accommodates a catenary curve (*thin solid line*) that passes by the extrados springing point F and is long enough to be tangent at the extrados point A at the crown. *Right*: The catenary curve, $h(x)$, is different than the two physically admissible minimum thrust-lines $\rho(\varphi)$ given by Eq. (6) and $\eta(x)$ given by Eq. (17)

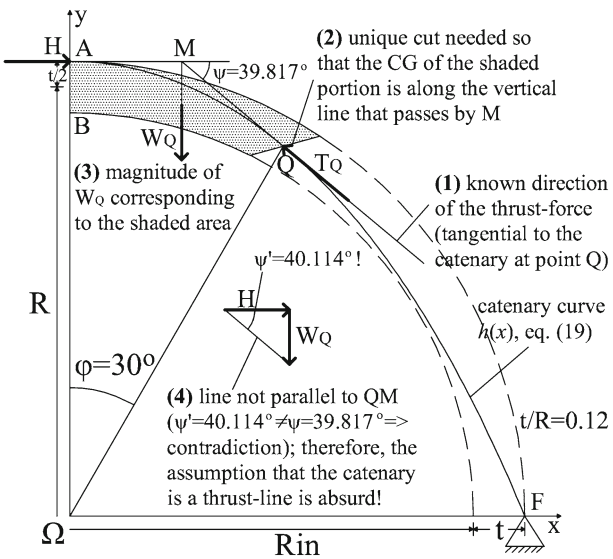


Fig. 7 Equilibrium check at $\varphi = 30°$ shows by reducing to the absurd that the catenary curve (the "hanging chain") appearing with a *thin solid line* is not a physically admissible minimum thrust-line of the semicircular arched monolith that is hinged at point F

With reference to Fig. 7, we concentrate on point Q on the catenary curve, that is, at some angle φ from the vertical axis y. Without any loss of generality, we take $\varphi = 30°$ as is shown in Fig. 7. Given that point Q belongs on a catenary curve; by definition, the thrust-force, T_Q, at point Q is along the known tangent of the catenary curve at point Q. This tangent at point Q crosses the line of action of the horizontal force H acting at the extrados point A of the crown at point M at an angle $\psi = 39.817°$.

At this stage, we have not assumed the direction of the cut which passes by point Q; nevertheless, this cut shall be such that the shaded portion of the arch ABQ to have its center of gravity on the vertical line that passes through point M. This requirement derives from the need that the thrust-forces H and T_Q together with the weight of the shaded portion of the arch W_Q satisfy moment equilibrium. Accordingly, the requirement that the line of action of W_Q to pass through point M uniquely defines the direction of the cut through point Q as shown in Fig. 7 (step 2).

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With the shape of the shaded portion of the arch being established from moment equilibrium, the magnitude of its weight W_Q is also uniquely defined and we are in the position to draw the force triangle that consists of the horizontal thrust H at the crown, the weight of the shaded portion of the arch W_Q, and the resultant thrust T_Q at point Q. When closing the triangle (see Fig. 7 (step 4)), the expected resultant thrust-force meets the horizontal thrust H at an angle $\psi = 40.114°$; therefore, it is not parallel to the tangent of the catenary at point Q (recall that the initial $\psi = 39.817°$); consequently, the assumption that the catenary curve is an admissible thrust-line which passes by points A and F is absurd. The above calculations have been worked with analytic geometry and have been confirmed with a commercially available, high precision drafting software [23].

The above result that was reached with reduction in the absurd (*reduction ad absurdum*) implies that when examining the problem of computing the minimum allowable thickness of a masonry arch, the catenary (Hooke's "hanging chain") is not a physically admissible thrust-line and shall be excluded from the family of admissible thrust-lines.

Figure 6 (right) zooms in the neighborhood isolated with the dashed parallelogram shown in Fig. 6 (left) and shows the differences between the catenary curve given by Eq. (19) and the two physically admissible minimum thrust-lines given by Eqs. (6) and (17).

3 Minimum thickness of a semicircular monolith with zero tensile and infinite compression strength

The problem of determining the minimum thickness of masonry arches has challenged the engineering community since the early eighteenth century ([9–12,19,22], and references reported therein).

3.1 Radial stereotomy and a polar coordinate system

3.1.1 Milankovitch's solution

Upon deriving the closed-form expression for the radius of curvature, $\rho(\varphi)$, of the minimum thrust-line given by Eq. (6), Milankovitch [7,8] computes for the first time the correct solution for the theoretical minimum thickness, t, of a circular arch with radius R. When the circular arch assumes its minimum thickness, the minimum thrust-line given by Eq. (6) also touches the intrados of the arch at some point K which defines the unknown rupture angle $\varphi_r = \mathrm{B\Omega K}$ as shown in Fig. 8. At the rupture angle, $\varphi_r = \mathrm{B\Omega K}$, the radius of the thrust-line $\rho(\varphi)$ given by Eq. (6) reaches a minimum ($\mathrm{d}\rho(\varphi)/\mathrm{d}\varphi = 0$) and for this value of φ_r, $\rho(\varphi_r) = R - t/2$. With these two constraints, Milankovitch [7,8] finds $\varphi_r = \mathrm{B\Omega K} = 54.484°$ and that the minimum allowable thickness is $t/R = 0.10748$.

3.1.2 Solution with a variational formulation

Milankovitch [7,8] computed the correct value of the minimum thickness of the arched monolith, $t/R = 0.10748$, by deriving the closed-form expression of the minimum thrust-line given by Eq. (6) in association with the information that when the circular monolith assumes its minimum thickness, the minimum thrust-line given by (6) also touches the intrados of the arch.

When the thickness of the arch is sufficiently reduced and the minimum thrust-line touches the intrados, the arch reaches a limit equilibrium state by developing a five-hinge symmetric mechanism [9]. Accordingly, points A, K, and F shown in Fig. 8 are imminent hinges of the arch at its limit equilibrium state. When assuming a rupture along the radial direction as shown in Fig. 9 (that is consistent with the radial stereotomy and the associated polar coordinate system adopted by Milankovitch [7,8]), the weight W_1 of the hinged portion of the arch ABK is

$$W_1 = tR\phi_\mathrm{r} \tag{20}$$

while the coordinates of the center of gravity of W_1 are

$$x_1 = R\frac{1-\cos\phi_\mathrm{r}}{\phi_\mathrm{r}}\left(1+\frac{1}{12}\frac{t^2}{R^2}\right) \tag{21a}$$

$$y_1 = R\frac{\sin\phi_\mathrm{r}}{\phi_\mathrm{r}}\left(1+\frac{1}{12}\frac{t^2}{R^2}\right) \tag{21b}$$

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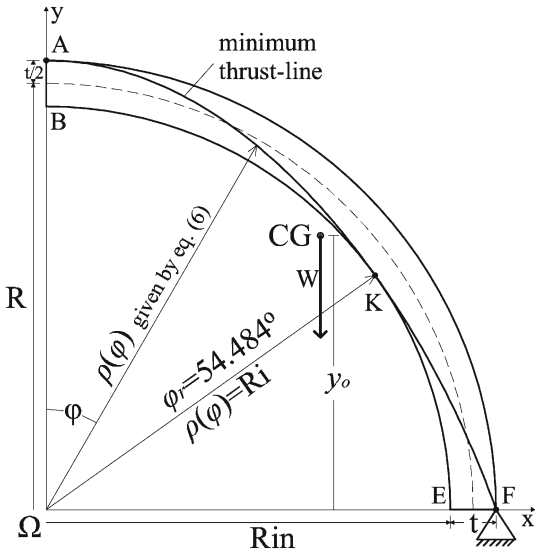


Fig. 8 Minimum thickness $t/R = 0.10748$ of a semicircular monolith as computed by Milankovitch [7,8] after adopting a radial stereotomy and a polar coordinate system together with the minimum thrust-line given by Eq. (6) which touches the intrados at the rupture angle $\varphi_r = 54.484°$. The center of gravity of the entire semi-arch is denoted with CG

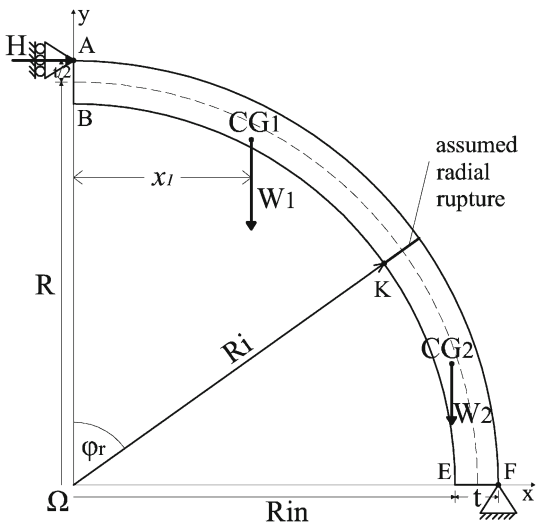


Fig. 9 Formation of a hinge mechanism by assuming that the rupture at the intrados hinge, K, happens along the radial direction. The centers of gravity of the two hinged portions of the semi-arch are denoted with CG_1 and CG_2

In Eqs. (20) and (21a,b), $\varphi_r = \mathrm{B\Omega K}$ is the unknown rupture angle of the arch.

With reference to Fig. 9, moment equilibrium of the top portion of the arch ABK about hinge K gives

$$H\left[R + \frac{t}{2} - \left(R - \frac{t}{2}\right)\cos\phi_r\right] = W_1\left[\left(R - \frac{t}{2}\right)\sin\phi_r - x_1\right] \tag{22}$$

while moment equilibrium of the entire half arch about hinge F gives Eq. (4).

Substitution of the expression of the minimum horizontal thrust H given by Eq. (4) and the expression of W_1 given by the Eq. (20) (that is for a radial cut) into the moment equilibrium equation given by Eq. (22) gives

$$\left[2\frac{t}{R}\left(\frac{t^2}{R^2} + 12\right) - 3\pi\left(\frac{t^2}{R^2} - 4\right)\right]\cos\phi_r = 3\pi\left(\frac{t}{R} + 2\right)^2 + 6\left(\frac{t^2}{R^2} - 4\right)\phi_r\sin\phi_r \tag{23}$$

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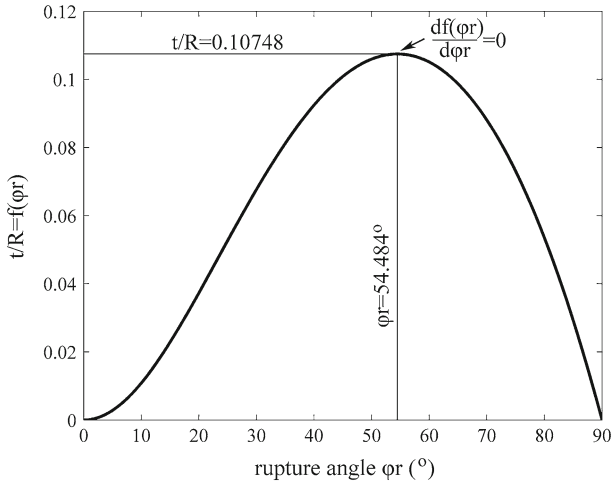


Fig. 10 Admissible values of t/R for a given rupture angle φ_r as they result from moment equilibrium of the hinged mechanism shown in Fig. 9. The principle of stationary potential energy ($\delta V(\varphi_r) = 0$) indicates that the hinged semicircular arch with a radial rupture is in equilibrium when $\mathrm{d}f(\varphi_r)/\mathrm{d}\varphi_r = 0$. This happens when $\varphi_r = 54.484°$ and the corresponding minimum thickness is $t/R = 0.10748$

The roots of Eq. (23) offer the admissible values of t/R for a given rupture angle φ_r. It can be rearranged as a cubic equation of t/R

$$a_1(\phi_r)\left(\frac{t}{R}\right)^3 + a_2(\phi_r)\left(\frac{t}{R}\right)^2 + a_3(\phi_r)\frac{t}{R} + a_4(\phi_r) = 0 \tag{24}$$

where the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are only functions of the rupture angle φ_r,

$$\begin{aligned} a_1(\phi_r) &= 2\cos(\phi_\mathrm{r}) \\ a_2(\phi_r) &= -3\left[\pi(1+\cos(\phi_\mathrm{r})) + 2\phi_\mathrm{r}\sin(\phi_\mathrm{r})\right] \\ a_3(\phi_r) &= -12(\pi - 2\cos(\phi_\mathrm{r})) \\ a_4(\phi_r) &= -12[\pi(1-\cos(\phi_\mathrm{r})) - 2\phi_\mathrm{r}\sin(\phi_\mathrm{r})]. \end{aligned} \tag{25}$$

Within the segment $0 \le \varphi_r \le \pi/2$, the discriminant,

$$\Delta = 18\alpha_1\alpha_2\alpha_3\alpha_4 - 4\alpha_2^3\alpha_4 + \alpha_2^2\alpha_3^2 - 4\alpha_1\alpha_3^3 - 27\alpha_1^2\alpha_4^2, \tag{26}$$

of Eq. (24) is positive; therefore, Eq. (24) has three real roots:

$$\frac{t}{R}|_j = f_j(\phi_r), \quad j \in \{1, 2, 3\} \tag{27}$$

The first root ($j = 1$) given by Eq. (27) yields negative real values of t/R, whereas the second root given by Eq. (27) ($j = 2$) yields real values of t/R larger than ten (10). Accordingly, both these roots are rejected and only the third root given by Eq. (27) ($j = 3$) is acceptable. Figure 10 plots the admissible values of $t/R = f(\varphi_r)$ for different values of the rupture angle φ_r as they result from the third root of Eq. (24).

Clearly, in addition to Eq. (24), there is a need for a second equation to solve for the two unknowns t/R and φ_r. This second equation is offered from the application of the principle of stationary potential energy. The fundamental concept of a variational formulation is that from among all geometrical admissible states (all admissible symmetric five-hinge mechanisms), a natural state—that is, a state which also satisfies equilibrium—can be detected by observing its behavior under infinitesimal geometrically admissible variations of the state. With reference to Fig. 9, the symmetric arch with minimum thickness is subjected only to its own weight (no external forces), while each of the two ruptured portions of the arch hinging at point K neither deform elastically nor dissipate energy at the hinges.

Adopting as a reference level the horizontal axis x ($y = 0$), the potential energy of the semicircular arch is

$$V = W \cdot y_o = R^3\frac{t}{R}\left(1 + \frac{1}{12}\frac{t^2}{R^2}\right) \tag{28}$$

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where $W = (\pi/2)tR$ is the weight and $y_o = R(12 + (t/R)^2)/6\pi$ is the vertical coordinate of the center of gravity of half the arch. When a radial rupture has been assumed the two unknowns, t/R and φ_r are related via moment equilibrium with the relation $t/R = f(\varphi_r)$ given by Eq. (27). Accordingly, the potential energy of the circular arch given by Eq. (28) can be expressed as a function of φ_r.

$$V(\phi_r) = R^3 f(\phi_r) \cdot \left(1 + \frac{1}{12} f(\phi_r)^2\right) \tag{29}$$

The structural system shown in Fig. 9 is a typical case where the only forces (weights) acting in the system are conservative and where the work of all forces is accounted by the potential energy $V(\varphi_r)$ given by Eq. (29). According to the principle of stationary potential energy, the geometrically admissible hinged mechanism shown in Fig. 9 is in an equilibrium state if and only if the total potential energy is stationary, i.e.,

$$\delta V(\phi_r) = \frac{\mathrm{d}V(\phi_r)}{\mathrm{d}\phi_r}\delta\phi_r = 0 \tag{30}$$

Substitution of Eq. (29) into Eq. (30) gives

$$\frac{\mathrm{d}}{\mathrm{d}\phi_r}\left[f(\phi_r) \cdot \left(1 + \frac{1}{12} f(\phi_r)^2\right)\right] = 0, \tag{31}$$

which simplifies to

$$\frac{\mathrm{d}f(\phi_r)}{\mathrm{d}\phi_r}\left[1 + \frac{1}{4}\, f(\phi_r)^2\right] = 0. \tag{32}$$

The quantity in brackets in Eq. (32) is always positive; therefore, Eq. (32) is satisfied when $\mathrm{d}f(\varphi_r)/\mathrm{d}\varphi_r = 0$. The remarkable result of Eq. (32) shows that the symmetric hinged arch is in an equilibrium state ($\delta V(\varphi_r) = 0$) if and only if the derivative of Eq. (27) vanishes.

$$\frac{\mathrm{d}f(\phi_r)}{\mathrm{d}\phi_r} = \frac{\mathrm{d}(t/R)}{\mathrm{d}\phi_r} = 0. \tag{33}$$

The solution of Eq. (33) offers the unknown location of the rupture angle $\varphi_r = 54.484°$ which is precisely the value computed by Milankovitch [7,8]. Substituting the value of the rupture angle $\varphi_r = 54.484°$ (that is for a rupture assumed along the radial direction) into the acceptable root of the moment equilibrium given by Eq. (27), one obtains the minimum thickness value $t/R = 0.10748$. Figure 10 shows that the maximum of Eq. (27) ($\mathrm{d}f(\varphi_r)/d\varphi_r = 0$) happens at $\varphi_r = 54.484°$ and that the maximum value of Eq. (27) is $t/R = 0.10748$.

Heyman [19] had pointed out that "the analysis may be made, although with much more complex algebraic manipulation, by writing the work equation for a small displacement of the mechanism" of Fig. 9. Without making reference to the principle of stationary potential energy given by Eq. (30), Heyman [19] explains that "No work is dissipated at a masonry hinge of the type considered here, so that the total work done by the loads acting on the arch (self-weight) must be zero. Thus, the work equation will give a single relationship between t/R and φ." This single relationship that Heyman was in search is Eq. (32) or merely Eq. (33) presented in this paper.

Equation (33) indicates that the correct rupture angle $\varphi_r = 54.484°$ and the minimum thickness $t/R = 0.10748$ can be derived by moment equilibrium alone by merely observing the graph of Eq. (27) appearing in Fig. 10. If the circular arched monolith is too thin ($t/R < 0.10748$). it will rupture at two distinct locations. A unique rupture (hinge) happens at the summit of the graph where $\mathrm{d}f(\varphi_r)/\mathrm{d}\varphi_r = 0$. This observation was "prophetically" stated by Heyman [19] who suggested in his own words "the value of φ_r must be chosen to maximize the value of t/R, and this condition leads eventually to the correct solution." Unfortunately, Heyman [19] did not proceed with the "work equation," rather he presented a slightly wrong answer for φ_r and t/R by taking the non-equality given by expression (9) as an equality (see also [12]).

Heyman's [19] "work-balance" concept was implemented by Oschendorf [24] who developed a trial-and-error procedure by selecting successive values of the rupture angle φ_r together with the corresponding values of t/R until the "work-balance" equation is satisfied. Oschendorf [24] assumed that the rupture of the arch at the intrados hinge is along the radial direction; therefore, for any assumed value of the rupture angle, φ_r, the weights and the coordinates of the center of gravity of the hinged segments of the arch are known. His trial-and-error procedure converged to the correct value of the rupture angle $\varphi_r = 54.5°$ and minimum thickness $t/R = 0.1075$ initially discovered by Milankovitch [7,8].

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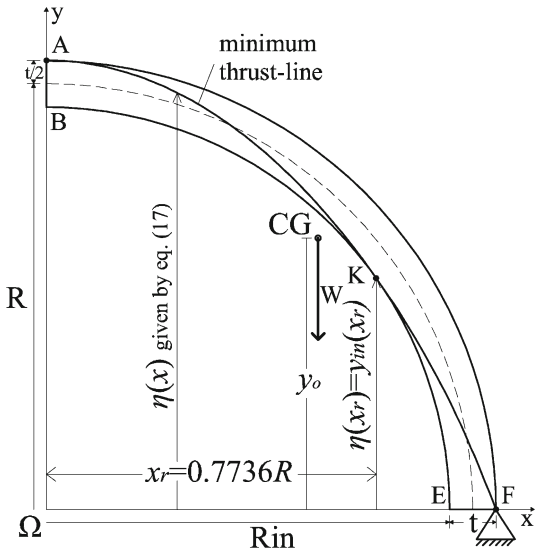


Fig. 11 Minimum thickness $t/R = 0.10946$ of a semicircular monolith computed with a geometric formulation after adopting a vertical stereotomy and a cartesian coordinate system together with the minimum thrust-line given by Eq. (17) which touches the intrados at $x_r = 0.7736R$ ($\varphi_r = \text{B}\Omega\text{K} = 54.923°$)

3.2 Vertical stereotomy and a cartesian coordinate system

3.2.1 A geometric solution

Equation (17) offers the ordinate of the minimum thrust-line of a semicircular arch that one computes after adopting a vertical stereotomy and a cartesian coordinate system. When the semicircular arch assumes its minimum thickness, the minimum thrust-line given by (17) also touches the intrados of the arch at some point K which defines the unknown rupture angle $\varphi_r = \text{B}\Omega\text{K}$. Since the minimum thrust-line given by Eq. (17) is different than the minimum thrust-line derived by Milankovitch [see Eq. (6)], it is expected that the location of the intrados hinge K derived with a cartesian coordinate system will be different than the location $\varphi_r = \text{B}\Omega\text{K} = 54.484°$ derived with a polar coordinate system.

So, with reference to Fig. 11, we are in search of the abscissa x_r for which

$$\eta(x_r) = y_{\text{in}}(x_r), \tag{34}$$

where $\eta(x)$ is the minimum thrust-line given by Eq. (17a) and $y_{\text{in}}(x)$ is given by Eq. (18b). In addition to the unknown rupture abscissa, x_r, both equations (17a) and (18b) contain the unknown quantity t/R. The second equation needed to determine t/R is that the thrust-line [not the thrust-force—although in this case, the line of action of the thrust-force is tangent to the thrust-line—see Eq. (10)] is tangent to the intrados at hinge K ($x = x_r$). Accordingly,

$$\frac{d\eta(x)}{dx}|_{x=x_r} = \frac{dy_{\text{in}}(x)}{dx}|_{x=x_r}. \tag{35}$$

From the two Eqs. (34) and (35), one computes $x_r = 0.7736R$ which corresponds to a rupture angle $\varphi_r = \text{B}\Omega\text{K} = 54.923°$ and a minimum thickness value $t/R = 0.10946$. The minimum allowable thickness, $t/R = 0.10946$, computed with a cartesian coordinate system (vertical cuts) is slightly larger than the value computed by Milankovitch [7,8] and with the aforementioned variational formulation presented in this work after adopting a radial cut ($t/R = 0.10748$); therefore, the adoption of a vertical stereotomy (cartesian coordinate system) yields a more conservative minimum thickness. This result is anticipated since according to Fig. 4 (right), in the vicinity of the imminent intrados hinge, the minimum thrust-line computed after adopting a vertical stereotomy [Eq. (17)] is closer to the intrados of the circular arch than the minimum thrust-line computed after adopting a radial stereotomy (polar coordinates) [Eq. (6)].

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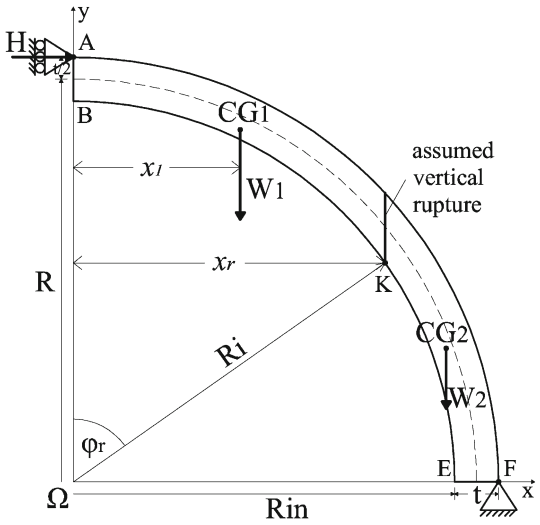


Fig. 12 Formation of a hinge mechanism by assuming that the rupture at the intrados hinge K happens along the vertical direction

3.2.2 Solution with a variational formulation

In this section, we consider again that the circular arch has reached its limit equilibrium state by developing a five-hinge symmetric mechanism. Accordingly, points A, K, and F shown in Fig. 11 are imminent hinges of the arch at its limit equilibrium state. We now assume that the rupture happens along the vertical direction as shown in Fig. 12 (that is consistent with a cartesian coordinate system). For a vertical rupture, the weight W_1 of the hinged portion of the semicircular arch ABK is

$$W_1 = R^2 \frac{1}{2}\left[\frac{x_r}{R}\left(\sqrt{c_1^2 - \frac{x_r^2}{R^2}} - \sqrt{c_2^2 - \frac{x_r^2}{R^2}}\right) + c_1^2 \arcsin\left(\frac{x_r}{Rc_1}\right) - c_2^2 \arcsin\left(\frac{x_r}{Rc_2}\right)\right] \tag{36}$$

while the coordinates of the center of gravity of W_1 are

$$x_1 = R\frac{2\frac{t}{R}\left(1 + \frac{1}{12}\frac{t^2}{R^2}\right) - \frac{2}{3}\left[\left(c_1^2 - \frac{x_r^2}{R^2}\right)^{3/2} - \left(c_2^2 - \frac{x_r^2}{R^2}\right)^{3/2}\right]}{\frac{x_r}{R}\left(\sqrt{c_1^2 - \frac{x_r^2}{R^2}} - \sqrt{c_2^2 - \frac{x_r^2}{R^2}}\right) + c_1^2 \arcsin\left(\frac{x_r}{Rc_1}\right) - c_2^2 \arcsin\left(\frac{x_r}{Rc_2}\right)} \tag{37a}$$

$$y_1 = R\frac{2\frac{x_r t}{R^2}}{\frac{x_r}{R}\left(\sqrt{c_1^2 - \frac{x_r^2}{R^2}} - \sqrt{c_2^2 - \frac{x_r^2}{R^2}}\right) + c_1^2 \arcsin\left(\frac{x_r}{Rc_1}\right) - c_2^2 \arcsin\left(\frac{x_r}{Rc_2}\right)} \tag{37b}$$

In Eqs. (36) and (37a,b), $x_r = Rc_2\sin(\varphi_r)$ is the unknown abscissa of the rupture of the arch, while c_1 and c_2 are given by Eq. (16).

With reference to Fig. 12, moment equilibrium of the top portion of the arch ABK about hinge K gives

$$H\left(R + \frac{t}{2} - y_{\text{in}}(x_r)\right) = W_1(x_r - x_1), \tag{38}$$

while moment equilibrium of the entire half arch about hinge F gives again Eq. (4). The expression of x_1 given by Eq. (37a) contains the terms $\arcsin(x_r/Rc_1)$ and $\arcsin(x_r/Rc_2)$; therefore, the roots of Eq. (38) cannot be expressed explicitly as was done in the case of Eq. (22). Substitution of the expression of the minimum horizontal force H at the crown given by Eq. (4) together with Eq. (18b), equation (38) becomes a transcendental equation that relates the unknown rupture location x_r/R with the unknown minimum thickness t/R. Numerical solution of Eq. (38) yields the graph shown in Fig. 13, which offers the admissible values of t/R

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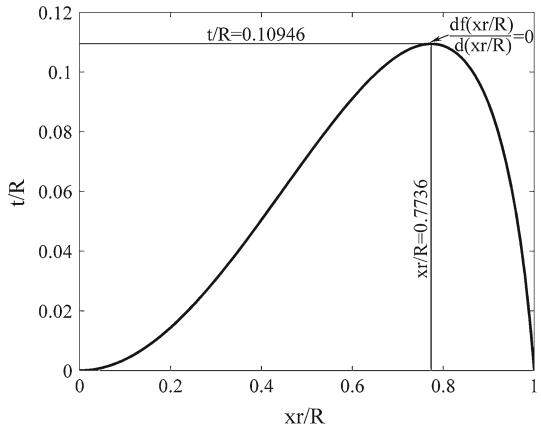


Fig. 13 Admissible values of t/R for a given rupture location x_r/R as they result from moment equilibrium of the hinged mechanism shown in Fig. 12. The principle of stationary potential energy ($\delta V(x_r/R) = 0$) indicates that the hinged semicircular arch with a vertical rupture is in equilibrium when $df(x_r/R)/d(x_r/R) = 0$. This happens when $x_r = 0.7736R$ and the corresponding minimum thickness is $t/R = 0.10946$

for a given value of the rupture location x_r/R. Figure 13 is the corresponding figure to Fig. 10 for the case where the rupture is now vertical rather than radial. Thus, the graph of Fig. 13 defines a functional relation

$$\frac{t}{R} = f\left(\frac{x_r}{R}\right) \tag{39}$$

between the admissible values of t/R and the potential rupture location x_r/R. With the functional relation $t/R = f(x_r/R)$ being established with the graph shown in Fig. 13, the potential energy of the circular arch given by Eq. (28) can be expressed as a function of x_r/R

$$V\left(\frac{x_r}{R}\right) = R^3 f\left(\frac{x_r}{R}\right) \cdot \left(1 + \frac{1}{12} f\left(\frac{x_r}{R}\right)^2\right). \tag{40}$$

According to the principle of stationary potential energy, the geometrically admissible hinged mechanism shown in Fig. 12 is in an equilibrium state if and only if the total potential energy is stationary, i.e., $\delta V(x_r/R) = 0$.

The calculus that follows is identical to the calculus that has been presented in the case of a polar coordinate system [Eqs. (30) to (33)]; therefore, the symmetric hinged arch with a vertical rupture at point K is in an equilibrium state ($\delta V(x_r/R) = 0$) if and only if

$$\frac{\mathrm{d}f\left(\frac{x_r}{R}\right)}{\mathrm{d}\frac{x_r}{R}} = \frac{\mathrm{d}\left(\frac{t}{R}\right)}{\mathrm{d}\frac{x_r}{R}} = 0. \tag{41}$$

Numerical processing of the graph appearing in Fig. 13 shows that the maximum of the graph ($\mathrm{d}f(x_r/R)/\mathrm{d}(x_r/R) = 0$) happens at $x_r = 0.7736R$ which corresponds to a rupture angle $\varphi_r = \mathrm{B\Omega K} = 54.923°$ and a minimum thickness value $t/R = 0.10946$. These values obtained with the principle of stationary potential energy are precisely the same values to those obtained with the geometric solution presented in the previous section in which Eqs. (34) and (35) have been used.

4 The educational and practical value of the "hanging chain"

Figure 7 shows that the catenary curve (the "hanging chain") that passes by the extrados springing point F and is long enough to be tangent to the extrados point A at the crown of a semicircular arch is not an admissible thrust-line since equilibrium checks on portions of the arch at arbitrary points along the catenary curve are not satisfied. Nevertheless, the catenary curve (the "hanging chain") is a neighboring line to the two physically admissible thrust-lines given by Eqs. (6) and (17) [see Fig. 6 (left)] and through the centuries, it has served as a remarkable educational and practical tool for the stability analysis of arches. Despite the aforementioned

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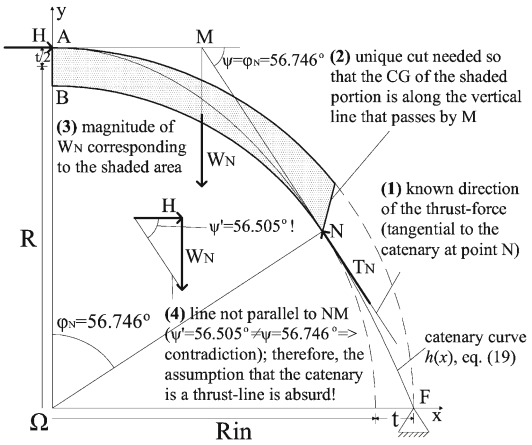


Fig. 14 Equilibrium check at $\varphi_N = 56.746°$ shows by reducing to the absurd that the catenary curve (the "hanging chain") which can only just be located within the semicircular arch is not a physically admissible thrust-line

minor equilibrium violations, the proportioning of a thin semicircular arch in the field will still be achieved by hanging a chain from the two springing points of the arch rather by plotting the admissible thrust-lines given by Eqs. (6) or (17).

Given the educational value and practical directness of the "hanging chain," it is worth computing what is the minimum thickness of a semicircular arch that is capable of accommodating the catenary curve given by Eq. (19) and shown in Fig. 6 (left). When the minimum thickness is reached, the catenary curve is tangent to the intrados of the arch at point N with abscissa x_N as shown in Fig. 14. At this location, N

$$h(x_N) = y_{in}(x_N) \tag{42}$$

and

$$\frac{dh(x)}{dx}|_{x=x_N} = \frac{dy_{in}(x)}{dx}|_{x=x_N}, \tag{43}$$

where $h(x)$ is the equation of the catenary curve that passes by the extrados points A and F given by Eq. (19) and $y_{in}(x)$ is the equation of the intrados of the semicircular arch given by Eq. (18b). Substitution of these expressions into Eqs. (42) and (43) yields

$$c_1\left[1 + a\left(1 - \cosh\left(\frac{1}{ac_1}\frac{x_N}{R}\right)\right)\right] - \sqrt{c_2^2 - \frac{x_N^2}{R^2}} = 0 \tag{44}$$

and

$$\sinh\left(\frac{1}{ac_1}\frac{x_N}{R}\right)\sqrt{c_2^2 - \frac{x_N^2}{R^2}} - \frac{x_N}{R} = 0 \tag{45}$$

where c_1 and c_2 are given by Eq. (16). The numerical solution of the system of Eqs. (44) and (45) yields the location $x_N/R = 0.7896$ ($\varphi_N = 56.746^o$) where the catenary curve touches the intrados of a semicircular arch and the corresponding thickness is $t/R = 0.11166$.

In the interest of completeness, Fig. 14 presents the steps of the equilibrium check of the portion of the arch ABN. Again, when closing the force triangle, the resultant thrust-force meets the horizontal thrust H at an angle $\psi = 56.505°$ which is different than the initial angle $\psi = \varphi_N = 56.746°$; therefore, the resultant thrust-force is not parallel to the tangent of the catenary curve at point N. Consequently, the assumption that the catenary curve is an admissible minimum thrust-line is absurd, and therefore, point N is not an imminent hinge.

The above result shows that the minimum thickness of a semicircular arch that is capable of accommodating the catenary curve ($t/R = 0.11166$) is even larger than the enhanced minimum thickness $t/R = 0.10946$

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Table 1 Minimum allowable thickness and rupture locations of a semicircular monolith with zero tensile strength

Radial stereotomy together with a polar coordinate system		Vertical stereotomy together with a cartesian coordinate system		Minimum thickness of a semicircular arch needed to accommodate a catenary curve
Rupture angle	Minimum thickness ($R = R_{in} + t/2$)	Rupture angle	Minimum thickness ($R = R_{in} + t/2$)	
$\varphi_r = \text{B}\Omega\text{K} = 54.484°$	$t/R = 0.10748$	$\varphi_r = \text{B}\Omega\text{K} = 54.923°$	$t/R = 0.10946$	$t/R = 0.11166 \approx 1/9$
Milankovitch [7,8]: geometric solution		This work: (a) geometric solution		This work: geometric solution
J. Ochsendorf [24]: Trial-and-error solution of the "work balance" equation		(b) Principle of stationary potential energy		
This work: Principle of stationary potential energy				The catenary curve is not a physically admissible thrust-line; therefore, the point touches the intrados of the arch ($\varphi_N = 56.746°$) is not an imminent hinge

computed in this paper after adopting a cartesian coordinate system. Consequently, the minimum thickness values adopted in practice by allowing the accommodation of R. Hooke's "hanging chain" work toward the safety of the arch.

The results derived in the last two sections for the minimum allowable thickness and the rupture location of a semicircular monolith with zero tensile strength subjected to its own weight are summarized in Table 1 together with the list of past publications which derived the correct results with various approaches.

5 Masonry arches with finite compression strength

The minimum allowable thickness values of the semicircular monolith summarized in Table 1 have been derived by allowing the thrust-line to be tangent at the intrados at point K and the extrados of the crown at point A (see Figs. 8, 11), therefore, by assuming that the masonry has "infinite" compressive strength.

When a finite compressive strength $\sigma_{max} < \infty$ is assumed, the maximum resultant thrust-force shall be at some sufficient distance away from the boundaries of the masonry arch. The three critical locations with stress concentration are the imminent hinges A, K, and F shown in Figs. 8 and 11 where the tangent forces have corresponding magnitudes $H \approx 0.4W < T_K < W$ (see Fig. 15).

When the resultant thrust-force is close to the boundary of the arch, it is reasonable to assume that the distribution of the internal compressive stress is triangular [25] so that the resultant thrust-force is at a distance, ε, away from either the extrados or intrados of the arch.

With reference to Fig. 15, the most intense force tangent to the boundary of the arch is the weight of the arch $W = (\pi/2)t_D R\gamma$ at the springing point F. Assuming a triangular stress distribution with maximum compressive stress σ_{max}, vertical equilibrium gives

$$\frac{1}{2}3\varepsilon\sigma_{max} = W = \frac{\pi}{2}t_D R\gamma, \tag{46}$$

where $t_D = t + 2\varepsilon$ is the "design" thickness of the arch and γ is the specific weight of the arch. From Eq. (46), one obtains

$$\frac{\varepsilon}{R} = \frac{\pi}{3}\frac{(t+2\varepsilon)\gamma}{\sigma_{max}} = \frac{1}{\frac{3}{\pi}\frac{\sigma_{max}}{t\gamma} - 2\frac{R}{t}} \tag{47}$$

Equation (47) indicates that the extra thickness ε/R is a function of the dimensionless strength of the arch $\sigma_{max}/t\gamma$ and the ratio t/R. Assuming a moderate value for the compressive strength of masonry, say $\sigma_{max} = 10\,\text{MPa}$, a specific weight $\gamma = 20\,\text{kN/m}^3$, and a thickness $t = 0.5\,\text{m}$, the dimensionless strength $\sigma_{max}/t\gamma = 1{,}000$, whereas from Table 1 $2R/t < 20$. Accordingly, even for moderate values of the compressive strength of masonry and relative large values of t, the additional thickness $\varepsilon/R \approx (\pi/3)\cdot(t\gamma/\sigma_{max}) < 0.001$, that is, more than two orders of magnitude less than the minimum values of t/R appearing in Table 1 which have been computed with the "infinite" strength assumption.

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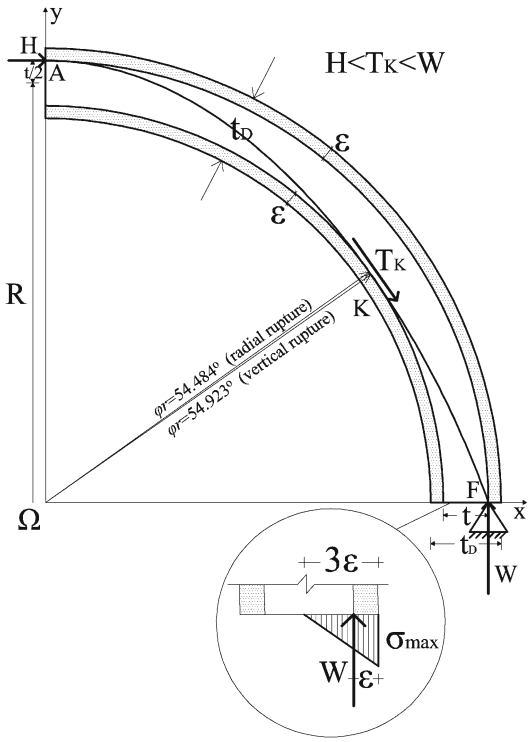


Fig. 15 Semicircular masonry arch with finite compression strength $\sigma_{\max}$ and design thickness $t_D = t + 2\varepsilon$

To this end, we note that the difference between the minimum thickness of the semicircular monolith $t/R = 0.11166$ that is needed to accommodate the "hanging chain" and the largest minimum allowable thickness $t/R = 0.10946$ which one obtains after adopting a cartesian coordinate system is 0.11166–0.10946 = 0.0022—a value that is in most practical cases larger than 2ε. Consequently, the minimum thickness values adopted in practice by allowing the accommodation of Hooke's "hanging chain" work toward the safety of the arch and as it is shown above, in most cases, this "design" thickness satisfies the limitations associated with the finite strength of masonry.

6 Comments on Heyman's 2009 "Scholia"

Heyman [12] published an article motivated from the need to reiterate that the line of action of the resultant thrust-force (funicular polygon) and the tangent to the thrust-line at the point of application of the thrust-force do not always coincide and that their mutual directions depend on the stereotomy exercised. In simple terms, Heyman [12] attempted to explain qualitatively the difference between the non-equality given by (9) and the equality given by (10)—a result that had been brought forward in a most rigorous manner by Milankovitch [7,8]. Part of the motivation for Heyman [12] to reiterate the difference between the non-equality (9) and the equality (10) was to explain the origin in the error that was committed in his 1969 publication [19] where he attempted to compute the minimum thickness of a semicircular arch. While the rupture at the intrados hinge K was assumed to happen along a radial direction (see Fig. 9), Heyman [19] incorrectly assumed that the line of action of the resultant thrust-force is tangent to the intrados of the arch—therefore tangent to the minimum thrust-line at point K; and this was the origin of the slight error in the calculation of his minimum thickness, $t/R = 0.106$, rather than the correct value $t/R = 0.1075$.

The last section of Heyman's 2009 article, entitled "SCHOLIA" presents a set of statement conclusions in an attempt to set straight the limit equilibrium analysis of the semicircular monolith.

His first statement concludes with "If Hooke's inverted hanging chain can only just be located within the arch, then the arch is at the point of collapse. In particular, the Author's 1969 analysis gives the correct solution to Couplet's problem if the arch is considered as a monolith." Unfortunately, both parts of the above 2009 statements are flawed. Regarding the first part of the statement in italics, Hooke's inverted "hanging chain" is

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not a physically admissible thrust-line (see Fig. 14); therefore, the intrados point N where the "hanging chain" touches the intrados is not an imminent hinge, and in order of the "hanging chain" to just be located within the arch, the thickness of the arch shall be $t/R = 0.11166$—a thickness that is even larger than the enhanced minimum thickness $t/R = 0.10946$ that one computes after adopting a cartesian coordinate system.

Regarding the second part of the statement in italics, Heyman's 1969 analysis ($t/R = 0.106$) remains unconservative given that the mathematically admissible solutions for the minimum thickness of the semicircular monolith are either $t/R = 0.10748$ [7,8] if one adopts a polar coordinate system, or $t/R = 0.10946$ if one adopts a cartesian coordinate system as was shown in this paper.

Heyman's [12] second statement continues: "As has been seen, the Author's 1969 analysis fails to achieve equilibrium for individual voussoirs (unless indeed the joints are all vertical)." Indeed, when the joints are all vertical the line of action of the resultant thrust-force is tangent to the thrust-line given by Eq. (17) at point K, as is dictated by Eq. (12); therefore, Heyman's [19] tangency condition holds. The flaw in Heyman's reasoning is that while he assumes a vertical joint (vertical cut), instead of expressing the moment equilibrium for the hinge mechanism shown in Fig. 12 (with a vertical rupture), he expresses the moment equilibrium for the hinge mechanism shown in Fig. 9 (with a radial rupture). For instance, when assuming a vertical rupture, the weight of the top hinged portion of the arch is given by Eq. (36), whereas Heyman [19] uses Eq. (20) which corresponds to a radial rupture. Simply stated, Heyman [12] mixes elements from two admissible hinge mechanisms; therefore; his solution $t/R = 0.106$ remains unconservative even if one assumes vertical joints. In fact, in the case of vertical joints, the correct solution for the minimum thickness of the semicircular monolith is $t/R = 0.10946$—a result that is larger than the minimum thickness $t/R = 0.10748$ computed by Milankovitch [7,8] and later confirmed by Ochsendorf [24] for the case of radial (Vitruvius) joints.

7 Conclusions

This paper revisits the limit equilibrium analysis of a monolithic semicircular arch with zero tensile strength and radius R. When the monolithic arch assumes its minimum thickness t, a symmetric five-hinge mechanism is imminent and at this state, any physically admissible thrust-line (line of resistance) shall pass by the extrados springing points and be tangent to the extrados at the center of the crown of the arch.

One such physically admissible limiting thrust-line was calculated by Milankovitch [7,8] who employed a radial stereotomy and adopted a polar coordinate system and by following a geometric formulation, he computed the minimum allowable thickness $t/R = 0.10748$ together with the location of the rupture of the arch $\varphi_r = 54.484°$.

This paper shows that the limiting thrust-line that was presented by Milankovitch [7,8] is not unique and that its profile depends on the stereotomy exercised. The adoption of a vertical stereotomy and a cartesian coordinate system yields a neighboring physically admissible thrust-line and a different, slightly higher value for the minimum allowable thickness ($t/R = 0.10946$) than the value computed by Milankovitch.

The minimum thickness value $t/R = 0.10946$ together with the associated rupture location $\varphi_r = 54.923°$ has been computed in this paper with a geometric formulation and confirmed with a variational formulation after assuming that the direction of the rupture happens along the vertical direction—an assumption that is consistent with the adoption of a vertical stereotomy in the geometric formulation.

The "counterintuitive" result of having two different yet neighboring theoretically correct answers for the limiting thrust-line and the minimum allowable thickness of semicircular arches depending on the stereotomy exercised is a consequence from the request to express the load path in a two-dimensional structure with finite thickness, t, with the thrust-line (line of resistance)—that is, a concept inherent to a one-dimensional structure (the "hanging chain").

Furthermore, the paper shows that the catenary (the "hanging chain") is not a physically admissible limiting thrust-line of the semicircular arch, although it is a neighboring line to the physically admissible lines derived: (a) by Milankovitch by employing a radial stereotomy and (b) in this paper after employing a vertical stereotomy.

The minimum thickness of a semicircular arch that is needed to accommodate the catenary curve is $t/R = 0.11166$, that is, a value even higher than the enhanced minimum thickness $t/R = 0.10946$ computed in this paper with cartesian coordinates. Consequently, the minimum thickness values adopted in practice with the use of Hooke's "hanging chain" works toward the safety of the arch. Finally, Heyman's [19] solution remains unconservative regardless the stereotomy exercised on the semicircular arch, even if one assumes vertical joints [12].

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